An equivalence result in the symmetric nonnegative inverse eigenvalue problem

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Joint work with Richard Ellard.

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NIEP and SNIEP

The nonnegative inverse eigenvalue problem (NIEP):
When is a list of n complex numbers

$$\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_n)$$

the spectrum of an $n \times n$ nonnegative matrix?

The symmetric NIEP (SNIEP):
When is a list of n real numbers

$$\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_n)$$

the spectrum of an $n \times n$ symmetric nonnegative matrix?

NIEP: Necessary conditions

The spectrum of a nonnegative matrix

$$\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_n)$$

satisfies the following conditions:

1. σ is closed under complex conjugation.
2. $s_k = \sum_{i=1}^{n} \lambda_i^k \geq 0$ for $k = 1, 2, \ldots$
3. The Perron eigenvalue

$$\lambda_1 = \max\{|\lambda_i|; \lambda_i \in \sigma\}$$

   lies in σ.

4. JLL-inequalities
   (Johnson (1981),Loewy and London (1978)):

$$s_k^m \leq n^{m-1} s_{km}$$

   for all $k, m = 1, 2, \ldots$

SNIEP: Additional necessary conditions

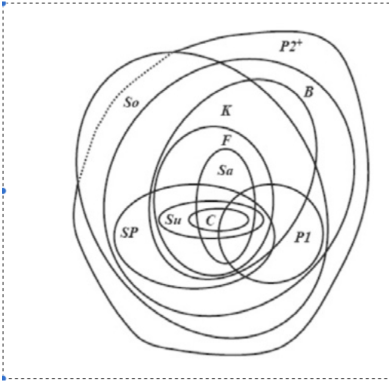
1. The spectrum of a symmetric nonnegative matrix is real.
2. (McDonald, Neumann (2000)) The spectrum of a 5×5 symmetric nonnegative matrix $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_5$ satisfies:

$$\lambda_1 + \lambda_3 + \lambda_4 \geq 0$$

Sufficient conditions

Several constructive techniques.
Suleimanova(1949), Perfect(1953), Ciarlet(1968), Kellog (1971), Salzmann (1972), Fiedler(1974), Borobia(1995), Soto(2003), Holtz(2005), etc.

Involved relations. Marijuan, Pisonero, Soto(2007)



Soules approach

Definition

Let R be an $n \times n$ real orthogonal matrix with columns $r_1, r_2, \ldots, r_n$. R is called a Soules matrix if r_1 is positive and for every diagonal matrix $\Lambda := \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$, the matrix $R\Lambda R^T$ is nonnegative.

Theorem (Soules(1983), Elsner, Nabben, Neumann(1998))

Let R be a Soules matrix and let $\Lambda := \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then the off-diagonal entries of the matrix $R\Lambda R^T$ are nonnegative.

The Soules Set $\mathcal{S}_n$

$$\begin{bmatrix} \frac{1}{2} & & & & & & & & & \\ \frac{1}{2} & & & & & & & & & \\ \frac{1}{2\sqrt{2}} & & & & & & & & & \\ \frac{\sqrt{3}}{4} & & & & & & & & & \\ \frac{\sqrt{3}}{4} & & & & & & & & & \end{bmatrix}$$

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$$\begin{tikzcd}[column sep=tiny]
 & & & & \{1,2,3,4,5\} \arrow[dl] \arrow[dr] & & & & \\
 & & & \{1,2\} \arrow[dl] & & \{3,4,5\} \arrow[dl] \arrow[dr] & & & \\
 & & \{1,2\} \arrow[dl] & & \{3\} \arrow[dl] & & \{4,5\} \arrow[dl] \arrow[dr] & & \\
 & \{1,2\} \arrow[dl] \arrow[dr] & & \{3\} \arrow[dr] & & \{4\} \arrow[dr] & & \{5\} \arrow[dr] & \\
\{1\} & & \{2\} & & \{3\} & & \{4\} & & \{5\}
\end{tikzcd}$$

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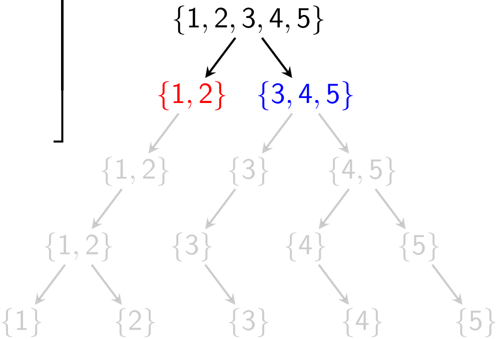

$$\begin{bmatrix} u \\ v \end{bmatrix} \to \frac{1}{\sqrt{||u||^2+||v||^2}} \begin{bmatrix} \frac{||v||}{||u||} u \\ -\frac{||u||}{||v||} v \end{bmatrix}$$

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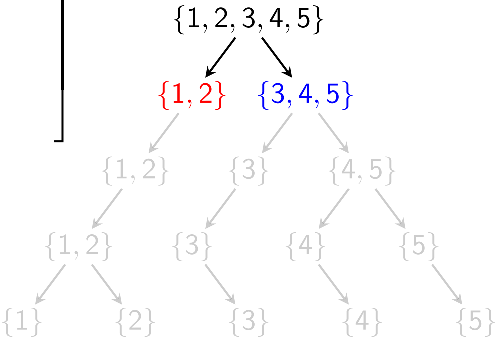

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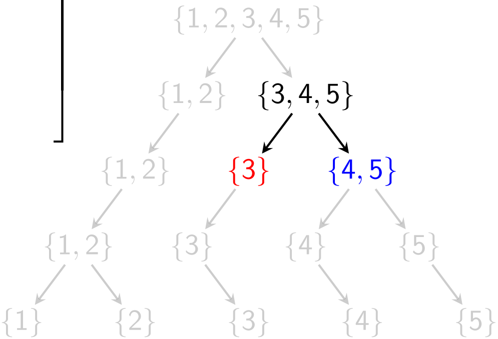

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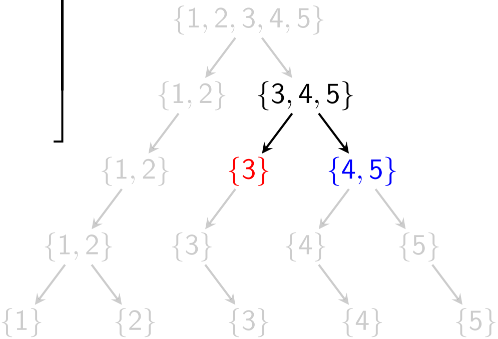
$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \color{red}{\frac{1}{2\sqrt{2}}} & -\frac{1}{2\sqrt{2}} & \color{red}{\frac{\sqrt{3}}{2}} & & \\ \color{blue}{\frac{\sqrt{3}}{4}} & -\frac{\sqrt{3}}{4} & \color{blue}{-\frac{1}{2\sqrt{2}}} & & \\ \color{blue}{\frac{\sqrt{3}}{4}} & -\frac{\sqrt{3}}{4} & \color{blue}{-\frac{1}{2\sqrt{2}}} & & \end{bmatrix}$$



$$\begin{bmatrix} \color{red}{u} \\ \color{blue}{v} \end{bmatrix} \to \frac{1}{\sqrt{||u||^2+||v||^2}} \begin{bmatrix} \color{red}{\frac{||v||}{||u||}u} \\ \color{blue}{-\frac{||u||}{||v||}v} \end{bmatrix}$$

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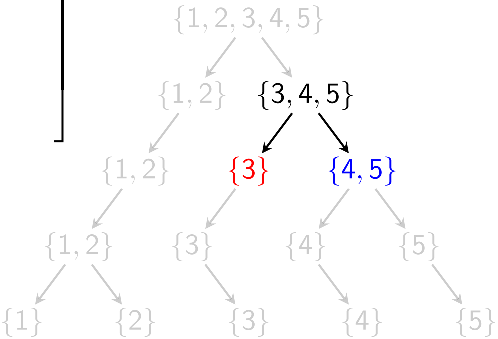
$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & & & \\ \frac{1}{2} & \frac{1}{2} & 0 & & & \\ \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2} & & & \\ \frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} & -\frac{1}{2\sqrt{2}} & & & \\ \frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} & -\frac{1}{2\sqrt{2}} & & & \end{bmatrix}$$



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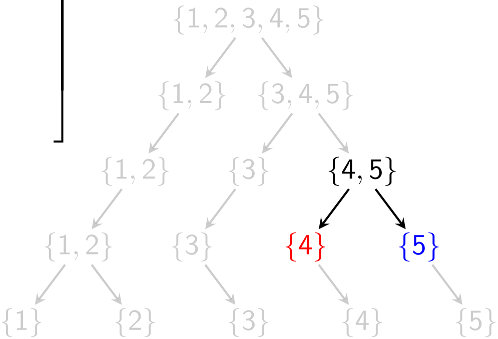

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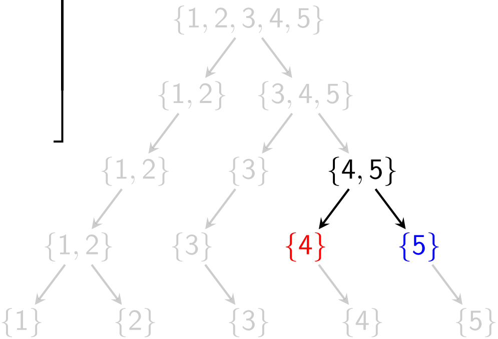

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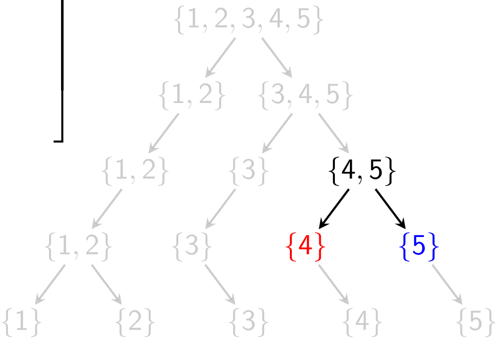

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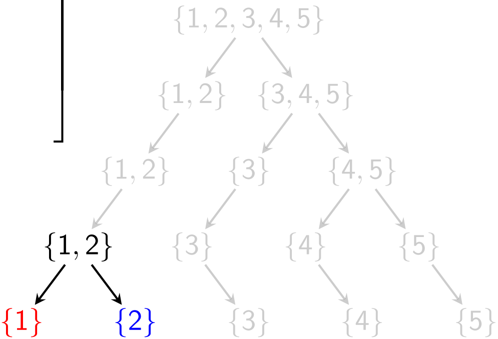

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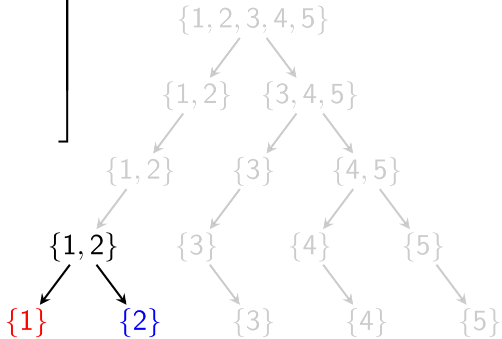

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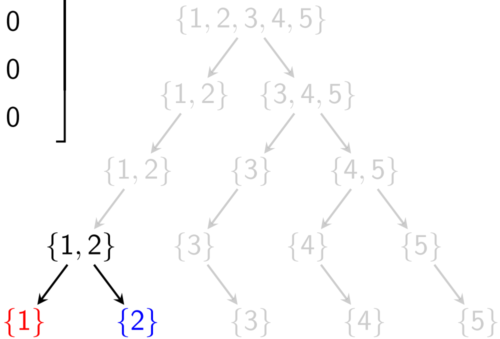

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The Soules Set $\mathcal{S}_n$

$$\Lambda = \mathrm{diag}(7, 5, -2, -4, -6)$$

$R =$

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} & -\frac{1}{2\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} & -\frac{1}{2\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

$R\Lambda R^T =$

$$\begin{bmatrix} 0 & 6 & \frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4} \\ 6 & 0 & \frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4} \\ \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & 0 & \sqrt{6} & \sqrt{6} \\ \frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4} & \sqrt{6} & 0 & 4 \\ \frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4} & \sqrt{6} & 4 & 0 \end{bmatrix}$$

A Recursive Approach to the SNIEP

Theorem (Š.(2004))

Let A be a nonnegative matrix with spectrum $(\mu_1, \mu_2, \ldots, \mu_n)$ and diagonal elements $(a_1, a_2, \ldots, a_{n-1}, c)$. Let B be a nonnegative matrix with Perron eigenvalue c, spectrum $(c, \lambda_2, \lambda_3, \ldots, \lambda_m)$ and diagonal elements $(b_1, b_2, \ldots, b_m)$. Then there exists a nonnegative matrix C with spectrum $(\mu_1, \mu_2, \ldots, \mu_n, \lambda_2, \lambda_3, \ldots, \lambda_m)$ and diagonal elements $(a_1, a_2, \ldots, a_{n-1}, b_1, b_2, \ldots, b_m)$.
Furthermore, if A and B are symmetric, then C may be chosen to be symmetric also.

Definition

We say $\sigma \in \mathcal{H}_n$ if it is possible to construct a nonnegative symmetric matrix with spectrum σ by repeated use of theorem above, starting with 2×2 matrices as building blocks.

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Furthermore, if A and B are symmetric, then C may be chosen to be symmetric also.

Definition

We say $\sigma \in \mathcal{H}_n(a_1, a_2, \ldots, a_n)$ if it is possible to construct a nonnegative symmetric matrix with spectrum σ and diagonal elements $(a_1, a_2, \ldots, a_n)$ by repeated use of the theorem, starting with 2×2 matrices as building blocks.

A Recursive Approach to the SNIEP

$$A = \begin{bmatrix} 0 & 6 & \frac{1}{\sqrt{2}} \\ 6 & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 6 \end{bmatrix} \text{ has spectrum } (7, 5, -6).$$

$$B = \begin{bmatrix} 0 & \sqrt{6} & \sqrt{6} \\ \sqrt{6} & 0 & 4 \\ \sqrt{6} & 4 & 0 \end{bmatrix} \text{ has spectrum } (6, -2, -4).$$

$$C = \begin{bmatrix} 0 & 6 & \frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4} \\ 6 & 0 & \frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4} \\ \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & 0 & \sqrt{6} & \sqrt{6} \\ \frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4} & \sqrt{6} & 0 & 4 \\ \frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4} & \sqrt{6} & 4 & 0 \end{bmatrix} \text{ has spectrum } (7, 5, -2, -4, -6).$$

A Recursive Approach to the SNIEP

$$A = X_1 \Lambda_1 X_1^T,\ X_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix},\ \Lambda_1 = \mathrm{diag}(7, 5, -6).$$

$$B = X_2 \Lambda_2 X_2^T,\ X_2 = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \sqrt{\frac{3}{8}} & -\frac{1}{2\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \sqrt{\frac{3}{8}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix},\ \Lambda_2 = \mathrm{diag}(6, -2, -4).$$

$$C = X \Lambda X^T,\ X = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & 0 & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} & 0 & -\frac{1}{2\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} & 0 & -\frac{1}{2\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix},$$

$$\Lambda = \mathrm{diag}(7, 5, -6, -2, -4).$$

C-realisability

Observation

If $(\lambda_1, \lambda_2, \ldots, \lambda_m)$ and $(\mu_1, \mu_2, \ldots, \mu_n)$ are realisable, then $(\lambda_1, \lambda_2, \ldots, \lambda_m, \mu_1, \mu_2, \ldots, \mu_n)$ is realisable.

Theorem

If $(\rho, \lambda_2, \lambda_3, \ldots, \lambda_n)$ is the spectrum of a (symmetric) nonnegative matrix with Perron eigenvalue ρ, then for all $\epsilon \geq 0$, $(\rho + \epsilon, \lambda_2, \lambda_3, \ldots, \lambda_n)$ is the spectrum of a (symmetric) nonnegative matrix also.

Theorem (Guo(1997))

If $(\rho, \lambda_2, \lambda_3, \ldots, \lambda_n)$ is the spectrum of a nonnegative matrix with Perron eigenvalue ρ, then for all $\epsilon \geq 0$, $(\rho + \epsilon, \lambda_2 \pm \epsilon, \lambda_3, \ldots, \lambda_n)$ is the spectrum of a nonnegative matrix also.

C-realisability

Definition (Borobia, Moro, Soto (2008))

A list of real numbers $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is called *C-realisable* if it may be obtained by starting with the n trivially realisable lists $(0), (0), \ldots, (0)$ and then using any of the previous three results any number of times in any order.

Example of C-realisability

$$(0),(0),(0),(0),(0),(0),(0),(0)$$
$$(0,0),(0,0),(0,0),(0,0)$$
$$(5,-5),(3,-3),(5,-5),(3,-3)$$
$$(5,3,-3,-5),(5,3,-3,-5)$$
$$({\color{red}7},3,{\color{red}-5},-5),(5,3,-3,-5)$$
$$(7,3,-5,-5),({\color{red}6},3,{\color{red}-4},-5)$$
$$(7,6,3,3,-4,-5,-5,-5)$$
$$({\color{red}8},6,3,3,{\color{red}-5},-5,-5,-5)$$

Equivalence Result

Theorem (Ellard, Š. 2016)

Let $\sigma := (\lambda_1, \lambda_2, \ldots, \lambda_n)$*, where* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ *and let* $a_1, a_2, \ldots, a_n \geq 0$*. Then the following are equivalent:*

(i) $\sigma \in \mathcal{S}_n$*;*

(ii) $\sigma \in \mathcal{H}_n$*;*

(iii) σ *is C-realisable;*

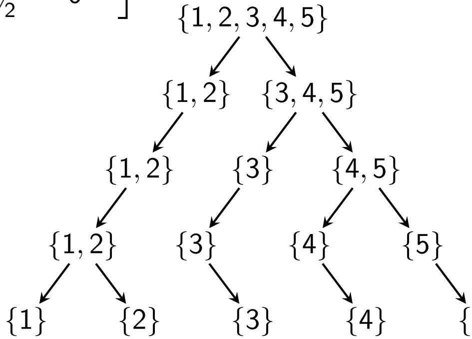
(iv) σ *satisfies* $\mathbb{S}_p$ *for some* p*.*

Furthermore, $\sigma \in \mathcal{H}_n(a_1, a_2, \ldots, a_n)$ *if and only if there exists an* $n \times n$ *Soules matrix* R *such that the matrix* $R \Lambda R^T$*—where* $\Lambda := \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$*—has diagonal elements* $(a_1, a_2, \ldots, a_n)$*.*

Equivalence Result: $\mathcal{S}_n \subseteq \mathcal{H}_n$

$R \Lambda R^T$ is nonnegative, where $\Lambda = \mathrm{diag}(7, 5, -2, -4, -6)$,

$$R = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} & -\frac{1}{2\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} & -\frac{1}{2\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}.$$



Equivalence Result: $\mathrm{S}_n \subseteq \mathcal{H}_n$

$$R = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} & -\frac{1}{2\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} & -\frac{1}{2\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

$$S_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \end{bmatrix}, \quad S_2 = \begin{bmatrix} \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} & -\frac{1}{2\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} & -\frac{1}{2\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix},$$

Equivalence Result: $\mathcal{S}_n \subseteq \mathcal{H}_n$

$$S_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \end{bmatrix}, \quad S_2 = \begin{bmatrix} \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} & -\frac{1}{2\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} & -\frac{1}{2\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix},$$

$$||u|| = \left|\left| \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}^T \right|\right| = \frac{1}{\sqrt{2}}, \quad ||v|| = \left|\left| \begin{bmatrix} \frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4} \end{bmatrix}^T \right|\right| = \frac{1}{\sqrt{2}},$$

$$c = ||v||^2\lambda_1 + ||u||^2\lambda_2 = 6$$

$$R_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}, \quad R_2 = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \sqrt{\frac{3}{8}} & -\frac{1}{2\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \sqrt{\frac{3}{8}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

Equivalence Result: $\mathcal{S}_n \subseteq \mathcal{H}_n$

$$(7; 5, -2, -4, -6) \in \mathcal{H}_5(0, 0, 0, 0, 0)$$

Equivalence Result: $\mathcal{S}_n \subseteq \mathcal{H}_n$

$$
\begin{tikzcd}[column sep=small]
 & (7;5,-2,-4,-6) \in \mathcal{H}_5(0,0,0,0,0) \arrow[dl] \arrow[dr] & \\
(7;5,-6) \in \mathcal{H}_3(0,0,\boxed{6}) & & (\boxed{6};-2,-4) \in \mathcal{H}_3(0,0,0)
\end{tikzcd}
$$

Equivalence Result: $\mathcal{S}_n \subseteq \mathcal{H}_n$

$$
\begin{tikzcd}[column sep=tiny]
 & & (7;5,-2,-4,-6) \in \mathcal{H}_5(0,0,0,0,0) \arrow[dl] \arrow[dr] & \\
 & (7;5,-6) \in \mathcal{H}_3(0,0,\boxed{6}) \arrow[dl] \arrow[dd] & & (\boxed{6};-2,-4) \in \mathcal{H}_3(0,0,0) \\
(7;5) \in \mathcal{H}_2(6,\boxed{6}) & & & \\
 & (\boxed{6};-6) \in \mathcal{H}_2(0,0) & &
\end{tikzcd}
$$

Equivalence Result: $\mathcal{S}_n \subseteq \mathcal{H}_n$

$$
\begin{tikzcd}[column sep=small]
 & & (7;5,-2,-4,-6) \in \mathcal{H}_5(0,0,0,0,0) \arrow[dl] \arrow[dr] & & \\
 & (7;5,-6) \in \mathcal{H}_3(0,0,\boxed{6}) \arrow[dl] \arrow[d] & & (\boxed{6};-2,-4) \in \mathcal{H}_3(0,0,0) \arrow[d] \arrow[dr] & \\
(7;5) \in \mathcal{H}_2(6,\boxed{6}) & (\boxed{6};-6) \in \mathcal{H}_2(0,0) & & (6;-2) \in \mathcal{H}_2(0,\boxed{4}) & (\boxed{4};-4) \in \mathcal{H}_2(0,0)
\end{tikzcd}
$$

Equivalence Result

$$\sigma \in \mathcal{S}_n$$

$$\Updownarrow$$

$$\sigma \in \mathcal{H}_n$$

Equivalence Result: $C - \mathrm{realizable} \subseteq \mathcal{H}_n$

Theorem (Ellard, Š. 2016)

Suppose $(\rho; \lambda_2, \lambda_3, \ldots, \lambda_n) \in \mathcal{H}_n(a_1, a_2, \ldots, a_n)$ *and* $\epsilon \geq 0$. *Then*

(i) $(\rho + \epsilon; \lambda_2 - \epsilon, \lambda_3, \lambda_4, \ldots, \lambda_n) \in \mathcal{H}_n(a_1, a_2, \ldots, a_n)$;

(ii) *there exist* $s, t \in \{1, 2, \ldots, n\}$, $s < t$, *such that*

$$(\rho + \epsilon, \lambda_2 + \epsilon, \lambda_3, \lambda_4, \ldots, \lambda_n) \in \mathcal{H}_n(a_1, \ldots, a_{s-1}, a_s + \epsilon, a_{s+1}, \ldots, a_{t-1}, a_t + \epsilon, a_{t+1}, \ldots, a_n).$$

In particular, if $(\rho; \lambda_2, \lambda_3, \ldots, \lambda_n) \in \mathcal{H}_n$, *then* $(\rho + \epsilon, \lambda_2 \pm \epsilon, \lambda_3, \lambda_4, \ldots, \lambda_n) \in \mathcal{H}_n$.

If σ is C-realisable, then $\sigma \in \mathcal{H}_n$

Equivalence Result

$$
\begin{tikzcd}
 & \sigma \in \mathcal{S}_n \arrow[d, Leftrightarrow] & \\
 & \sigma \in \mathcal{H}_n \arrow[dr, Rightarrow] & \\
\sigma \text{ is C-realisable} \arrow[ur, Rightarrow] & & \sigma \text{ satisfies } \mathbb{S}_p \arrow[ll, Rightarrow]
\end{tikzcd}
$$

Reducing the Spectral Gap

Theorem (Ellard, Š. 2016)

If $(\lambda_1; \lambda_2, \lambda_3, \ldots, \lambda_n) \in \mathcal{H}_n(a_1, a_2, \ldots, a_n)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, *then there exist* $\epsilon \geq 0$ *and partitions*

$$\{3, 4, \ldots, n\} = \{p_1, p_2, \ldots, p_{l-1}\} \cup \{q_1, q_2, \ldots, q_{n-l-1}\}$$

and

$$\{1, 2, \ldots, n\} = \{r_1, r_2, \ldots, r_l\} \cup \{s_1, s_2, \ldots, s_{n-l}\}$$

such that

$$(\lambda_1 - \epsilon; \lambda_{p_1}, \lambda_{p_2}, \ldots, \lambda_{p_{l-1}}) \in \mathcal{H}_l(a_{r_1}, a_{r_2}, \ldots, a_{r_l})$$

and

$$(\lambda_2 + \epsilon; \lambda_{q_1}, \lambda_{q_2}, \ldots, \lambda_{q_{n-l-1}}) \in \mathcal{H}_{n-l}(a_{s_1}, a_{s_2}, \ldots, a_{s_{n-l}}).$$

e.g. $(7; 5, -2, -4, -6) \in \mathcal{H}_5(0, 0, 0, 0, 0)$

Reducing the Spectral Gap

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If $(\lambda_1; \lambda_2, \lambda_3, \ldots, \lambda_n) \in \mathcal{H}_n(a_1, a_2, \ldots, a_n)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, *then there exist* $\epsilon \geq 0$ *and partitions*

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such that

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and

$$(\lambda_2 + \epsilon; \lambda_{q_1}, \lambda_{q_2}, \ldots, \lambda_{q_{n-l-1}}) \in \mathcal{H}_{n-l}(a_{s_1}, a_{s_2}, \ldots, a_{s_{n-l}}).$$

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Reducing the Spectral Gap

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and

$$\{1, 2, \ldots, n\} = \{r_1, r_2, \ldots, r_l\} \cup \{s_1, s_2, \ldots, s_{n-l}\}$$

such that

$$(\lambda_1 - \epsilon; \lambda_{p_1}, \lambda_{p_2}, \ldots, \lambda_{p_{l-1}}) \in \mathcal{H}_l(a_{r_1}, a_{r_2}, \ldots, a_{r_l})$$

and

$$(\lambda_2 + \epsilon; \lambda_{q_1}, \lambda_{q_2}, \ldots, \lambda_{q_{n-l-1}}) \in \mathcal{H}_{n-l}(a_{s_1}, a_{s_2}, \ldots, a_{s_{n-l}}).$$

e.g. $(6; 6, -2, -4, -6) \in \mathcal{H}_5(0, 0, 0, 0, 0)$

Reducing the Spectral Gap

Theorem (Ellard, Š. 2016)

If $(\lambda_1; \lambda_2, \lambda_3, \dots, \lambda_n) \in \mathcal{H}_n(a_1, a_2, \dots, a_n)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, *then there exist* $\epsilon \geq 0$ *and partitions*

$$\{3, 4, \dots, n\} = \{p_1, p_2, \dots, p_{l-1}\} \cup \{q_1, q_2, \dots, q_{n-l-1}\}$$

and

$$\{1, 2, \dots, n\} = \{r_1, r_2, \dots, r_l\} \cup \{s_1, s_2, \dots, s_{n-l}\}$$

such that

$$(\lambda_1 - \epsilon; \lambda_{p_1}, \lambda_{p_2}, \dots, \lambda_{p_{l-1}}) \in \mathcal{H}_l(a_{r_1}, a_{r_2}, \dots, a_{r_l})$$

and

$$(\lambda_2 + \epsilon; \lambda_{q_1}, \lambda_{q_2}, \dots, \lambda_{q_{n-l-1}}) \in \mathcal{H}_{n-l}(a_{s_1}, a_{s_2}, \dots, a_{s_{n-l}}).$$

e.g. $(6; 6, -2, -4, -6) \in \mathcal{H}_5(0, 0, 0, 0, 0)$

Effect of Adding Zeros

Theorem (Ellard, Š. 2016)

If

$$(\lambda_1; \lambda_2, \ldots, \lambda_n, 0) \in \mathcal{H}_{n+1}(a_1, a_2, \ldots, a_{n+1}),$$

then there exist $s, t \in \{1, 2, \ldots, n+1\}$*,* $s < t$*, such that*

$$(\lambda_1; \lambda_2, \ldots, \lambda_n) \in \mathcal{H}_n(a_1, \ldots, a_{s-1}, a_{s+1}, \ldots, a_{t-1}, a_{t+1}, \ldots, a_{n+1}, a_s + a_t).$$

In particular, if $(\lambda_1, \lambda_2, \ldots, \lambda_n, 0) \in \mathcal{H}_{n+1}$*, then* $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathcal{H}_n$*.*

Symmetrically Realisable Lists Outside $\mathcal{S}_n$?

Minimum t such that

$$(3+t, 3-t, -2, -2, -2)$$

is symmetrically realisable?

Symmetrically Realisable Lists Outside $\mathcal{H}_n$?

Minimum t such that

$$(3+t, 3-t, -2, -2, -2)$$

is symmetrically realisable is $t = 1$ (McDonald and Neumann, 2000).

Symmetrically Realisable Lists Outside $\mathcal{H}_n$?

Minimum t such that

$$(3+t, 3-t, \quad, -2, -2, -2)$$

is symmetrically realisable is

Symmetrically Realisable Lists Outside $\mathcal{H}_n$?

Minimum t such that

$$(3+t, 3-t, 0, -2, -2, -2)$$

is symmetrically realisable is

Symmetrically Realisable Lists Outside $\mathcal{H}_n$?

Minimum t such that

$$(3+t, 3-t, 0, -2, -2, -2)$$

is symmetrically realisable is $\leq \frac{1}{3}$ (Laffey and Šmigoc, 2007).

$$\begin{bmatrix} 2 & \sqrt{\frac{8}{3}} & 0 & 0 \\ \sqrt{\frac{8}{3}} & 0 & \frac{4}{3} & 0 \\ 0 & \frac{4}{3} & 0 & \sqrt{\frac{8}{3}} \\ 0 & 0 & \sqrt{\frac{8}{3}} & 2 \end{bmatrix} \text{ has spectrum } \left(\tfrac{10}{3}, \tfrac{8}{3}, 0, -2\right).$$

Symmetrically Realisable Lists Outside $\mathcal{S}_n$?

Lemma (Ellard, Š. 2017)

Let $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$. *Then* $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ *is the spectrum of a nonnegative symmetric matrix of the form*

$$\begin{bmatrix} a & b_1 & 0 & 0 \\ b_1 & 0 & b_2 & 0 \\ 0 & b_2 & 0 & b_3 \\ 0 & 0 & b_3 & a \end{bmatrix}$$

if and only if the following conditions are satisfied:

$$\lambda_1 + \lambda_2 + \lambda_2 + \lambda_4 \geq 0,$$
$$\lambda_2 \geq \lambda_1 + \lambda_3 + \lambda_4,$$
$$\lambda_1^2 - \lambda_2^2 + \lambda_3^2 - \lambda_4^2 \geq 0.$$

Symmetrically Realisable Lists Outside $\mathcal{S}_n$?

Minimum t such that $(4+t, 4, 4-t, 0, -3, -3, -3, -3)$ is symmetrically realisable?

In $\mathcal{H}_8$: $t = 2$.
Using 2×2 matrices plus lemma: $t = 1$.

—

Minimum t such that $(5+t, 5, 5, 5-t, 0, -4, -4, -4, -4, -4)$ is symmetrically realisable?

In $\mathcal{H}_{10}$: $t = 6$.
Using 2×2 matrices plus lemma: $t = 2$.