The Birkhoff–James and Roberts orthogonality in C^*-algebras

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In a normed linear space $(X, \|\cdot\|)$, we say that

◇ $x \in X$ is **orthogonal** to $y \in X$ **in the Birkhoff–James sense**, $x \perp_{BJ} y$, if

$$\|x\| \leq \|x + \lambda y\|, \quad \forall \lambda \in \mathbb{C}.$$

◇ $x \in X$ and $y \in X$ are **Roberts orthogonal**, $x \perp_R y$, if

$$\|x + \lambda y\| = \|x - \lambda y\|, \quad \forall \lambda \in \mathbb{C}.$$

- $x \perp_{BJ} y \Leftrightarrow \|x\| = \min_{\lambda \in \mathbb{C}} \|x + \lambda y\| \Leftrightarrow \|x\| = \mathrm{d}(x, \mathbb{C}y)$.
- The Birkhoff–James orthogonality in normed linear spaces
  - is *nondegenerate*: $(x \perp_{BJ} x \Leftrightarrow x = 0)$;
  - is *homogeneous*: $(x \perp_{BJ} y \Rightarrow \lambda x \perp_{BJ} \mu y, \ \forall \lambda, \mu \in \mathbb{C})$;
  - *is not symmetric*: $(x \perp_{BJ} y \not\Rightarrow y \perp_{BJ} x)$;
  - *is not additive*: $\big((x \perp_{BJ} y \text{ and } x \perp_{BJ} z) \not\Rightarrow x \perp_{BJ} (y + z)\big)$.
- For every two elements $x, y \in X$ there exists $\lambda \in \mathbb{C}$ such that $x \perp_{BJ} (\lambda x + y)$.

Roberts orthogonality in normed linear spaces

- $x \perp_R y$ if $\|x + \lambda y\| = \|x - \lambda y\|$ for all $\lambda \in \mathbb{C}$.
- Roberts orthogonality in normed linear spaces
  - is *nondegenerate*: $(x \perp_R x \Leftrightarrow x = 0)$;
  - is *homogeneous*: $(x \perp_R y \Rightarrow \lambda x \perp_R \mu y, \ \forall \lambda, \mu \in \mathbb{C})$;
  - *is symmetric*: $(x \perp_R y \Leftrightarrow y \perp_R x)$;
  - *is not additive*: $\big((x \perp_R y \textit{ and } x \perp_R z) \not\Rightarrow x \perp_R (y+z)\big)$.
- R-orthogonality does not have the existence property.
- $x \perp_R y \Rightarrow x \perp_{BJ} y$.
- If $(X, (\cdot, \cdot))$ is an inner product space, then $x \perp_R y \Leftrightarrow x \perp_{BJ} y \Leftrightarrow (x, y) = 0$.

A C^*-**algebra** $\mathcal{A}$ is a Banach $*$-algebra with the norm satisfying the C^*-condition $\|a^*a\| = \|a\|^2$ for all $a \in \mathcal{A}$.

Gelfand–Naimark theorem

Every C^*-algebra $\mathcal{A}$ can be regarded as a C^*-subalgebra of $B(H)$ for some Hilbert space H, that is, there exist a Hilbert space H and a faithful (injective) $*$-homomorphism $\varphi : \mathcal{A} \longrightarrow B(H)$.

$(H, (\cdot,\cdot))$ - Hilbert space

$B(H)$ - algebra of all bounded linear operators on H

THEOREM (R. BHATIA, P. ŠEMRL, 1999)

Let $A, B \in B(H)$.

- *$A \perp_B B$ if and only if there is a sequence (x_n) in H, $\|x_n\| = 1$, such that $\lim\limits_{n\to\infty} \|Ax_n\| = \|A\|$ and $\lim\limits_{n\to\infty} (Ax_n, Bx_n) = 0$.*
- *If $\dim H < \infty$, then $A \perp_B B$ if and only if there is a unit vector x in H such that $\|Ax\| = \|A\|$ and $(Ax, Bx) = 0$.*

$S(\mathcal{A})$ – set of all states of $\mathcal{A}$ (i.e., the set of all positive norm one linear functionals of $\mathcal{A}$)

Theorem (Lj. Arambašić, R. R., 2012)

Let $\mathcal{A}$ be a C^*-algebra, and $a, b \in \mathcal{A}$. Then $a \perp_{BJ} b$ if and only if there is $\varphi \in S(\mathcal{A})$ such that $\varphi(a^*a) = \|a\|^2$ and $\varphi(a^*b) = 0$.

$\mathcal{A}$ – C^*-algebra with unit e

$V(a)$ – numerical range of $a \in \mathcal{A}$
$V(a) = \{\varphi(a) : \varphi \text{ is a state of } \mathcal{A}\}$

$V_{max}(a)$ – maximal numerical range of $a \in \mathcal{A}$
$V_{max}(a) = \{\varphi(a) : \varphi \text{ is a state of } \mathcal{A},\ \varphi(a^*a) = \|a\|^2\}$

(J.G. Stampfli, J.P. Williams, 1968, 1970)

$e \perp_{BJ} a \Leftrightarrow 0 \in V(a)$;

$a \perp_{BJ} e \Leftrightarrow 0 \in V_{max}(a)$.

Roberts orthogonality in C^*-algebras

$\mathcal{A}$ – C^*-algebra with unit e

Let a be a **self-adjoint** element of $\mathcal{A}$. Suppose that $a \perp_R e$.

$$V(a) = \mathrm{conv}(\sigma(a)) = [\alpha, \beta] \subseteq [-\|a\|, \|a\|],$$

where $\alpha = -\|a\|$ or $\beta = \|a\|$. Then for every $\lambda \in \mathbb{C}$

$$\max\{|\alpha+\lambda|, |\beta+\lambda|\} = \|a+\lambda e\| = \|a-\lambda e\| = \max\{|\alpha-\lambda|, |\beta-\lambda|\},$$

from which it follows that $\alpha = -\beta$. Thus, $V(a) = [-\|a\|, \|a\|]$, and therefore $V(a) = -V(a)$. The converse is obvious.

If $a \in \mathcal{A}$ is self-adjoint, then

$$a \perp_R e \Leftrightarrow V(a) = -V(a) \Leftrightarrow \pm\|a\| \in \sigma(a).$$

Proposition (Lj. Arambašić, T. Berić, R. R.)

Let $\mathcal{A}$ be a C^*-algebra with the unit e, and $a \in \mathcal{A}$. If $a \perp_R e$, then $V(a) = -V(a)$.

Let a be a **normal** element of $\mathcal{A}$. Then

$$\|a\| = w(a) = \max\{|z| : z \in V(a)\}.$$

Since $a + \lambda e$ is normal for every $\lambda \in \mathbb{C}$,

$$a \perp_R e \Leftrightarrow w(a + \lambda e) = w(a - \lambda e), \quad \forall \lambda \in \mathbb{C}.$$

Proposition

Let $\mathcal{A}$ be a C^*-algebra with the unit e. If $a \in \mathcal{A}$ is normal, then

$$a \perp_R e \Leftrightarrow V(a) = -V(a).$$

In general, $V(a) = -V(a)$ does not imply $a \perp_R e$.

Example (M.-T. Chien, B. S. Tam)

Let $\mathcal{A} = \mathbb{M}_4(\mathbb{C}) = B(\mathbb{C}^4)$. Then

$$V(A) = W(A) = \{(Ax, x) : x \in \mathbb{C}^4, \|x\| = 1\}.$$

The numerical range $W(A)$ of the matrix

$$A = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is a circular disk centered at the origin, so $W(A) = -W(A)$. $2.6918 = \|A + I\| \neq \|A - I\| = 2.7578$, so A is not Roberts orthogonal to the identity operator I.

Let $a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. There is $\varphi \in S(\mathcal{A})$ such that

$$\begin{aligned} \|a+\lambda e\|^2 &= \|(a+\lambda e)^*(a+\lambda e)\| \\ &= w((a+\lambda e)^*(a+\lambda e)) \\ &= \varphi((a+\lambda e)^*(a+\lambda e)) \\ &= \varphi(a^*a) + 2\mathrm{Re}(\bar{\lambda}\varphi(a)) + |\lambda|^2 \\ &\geq \psi(a^*a) + 2\mathrm{Re}(\bar{\lambda}\psi(a)) + |\lambda|^2 \\ &= \psi((a+\lambda e)^*(a+\lambda e)), \qquad \forall \psi \in S(\mathcal{A}). \end{aligned}$$

Let us denote $\mu := \varphi(a)$. Then

$$\varphi(a^*a) \geq \psi(a^*a) \text{ for every } \psi \in S(\mathcal{A}) \text{ such that } \psi(a) = \mu.$$

Therefore,

$$\varphi(a^*a) = \max \mathcal{L}_\mu(a),$$

where

$$\mathcal{L}_\mu(a) = \{\psi(a^*a) : \psi \in S(\mathcal{A}), \psi(a) = \mu\}.$$

The **Davis–Wielandt shell** of $a \in \mathcal{A}$ is defined as the set

$$DV(a) = \{(\varphi(a), \varphi(a^*a)) : \varphi \in S(\mathcal{A})\}.$$

Since $S(\mathcal{A})$ is a weak*-compact and convex subset of $\mathcal{A}'$, and the map $\varphi \mapsto (\varphi(a), \varphi(a^*a))$ is weak*-continuous on $\mathcal{A}'$, we conclude that $DV(a)$ is a compact convex subset of $\mathbb{C} \times \mathbb{R}$.

The **upper boundary** of $DV(a)$ is the set

$$DV_{ub}(a) = \{(\mu, r) \in DV(a) : r = \max \mathcal{L}_\mu(a)\},$$

where

$$\mathcal{L}_\mu(a) = \{\varphi(a^*a) : \varphi \in S(\mathcal{A}), \varphi(a) = \mu\}.$$

Note that $(\varphi(a), \varphi(a^*a)) \in DV_{ub}(a)$ if
$\|a + \lambda e\|^2 = \varphi((a + \lambda e)^*(a + \lambda e))$ for some $\lambda \in \mathbb{C}$.

Theorem (Lj. Arambašić, T. Berić, R. R.)

Let $\mathcal{A}$ be a C^*-algebra with the unit e. For $a \in \mathcal{A}$ the following conditions are mutually equivalent:

(i) $a \perp_R e$,

(ii) $DV_{ub}(a) = DV_{ub}(-a)$.

Let $\mathcal{A} = B(H)$. The **Davis–Wielandt shell** of $A \in B(H)$ is defined as the set

$$DW(A) = \{((Ax, x), (A^*Ax, x)) : x \in H, \|x\| = 1\}.$$

Since $A = \operatorname{Re} A + i \operatorname{Im} A$, identifying $\mathbb{C} \times \mathbb{R}$ with $\mathbb{R}^3$, we have

$$DW(A) = ((\operatorname{Re} A)x, x), ((\operatorname{Im} A)x, x), (A^*Ax, x)) : x \in H, \|x\| = 1\},$$

which is a *joint numerical range* of self-adjoint operators $\operatorname{Re} A, \operatorname{Im} A$ and A^*A.

- $\diamond$ $DW(A)$ is compact if $\dim H < \infty$.
- $\diamond$ $DW(A)$ is convex if $\dim H \geq 3$.
- $\diamond$ $DV(A) = \overline{\operatorname{conv}(DW(A))}$.

Therefore

$$\begin{aligned} DV(A) &= \overline{DW(A)} \quad \text{if H is infinite-dimensional,} \\ DV(A) &= DW(A) \quad \text{if } 3 \leq \dim H < \infty. \end{aligned}$$

If H is infinite-dimensional, then

$$DV_{ub}(A) = \{(\mu, r) \in \overline{DW(A)} : \ r = \max \mathcal{L}_\mu(A)\},$$

where

$$\mathcal{L}_\mu(A) = \{ \lim_{n\to\infty} (A^*Ax_n, x_n) : \ x_n \in H, \|x_n\| = 1, \lim_{n\to\infty} (Ax_n, x_n) = \mu\}.$$

If $3 \leq \dim H < \infty$, then

$$DV_{ub}(A) = \{(\mu, r) \in DW(A) : \ r = \max \mathcal{L}_\mu(A)\},$$

where

$$\mathcal{L}_\mu(A) = \{(A^*Ax, x) : \ x \in H, \|x\| = 1, (Ax, x) = \mu\}.$$

THEOREM

Let $A \in B(H)$, $\dim H \geq 3$. *Then the following conditions are mutually equivalent*:

(I) $A \perp_R I$,

(II) $DV_{ub}(A) = DV_{ub}(-A)$.

Let $A \in B(H)$, $\dim H = 2$.

Let α and β be eigenvalues of A. The numerical range $W(A)$ is an elliptical disc (possibly degenerate) centered at $\frac{1}{2}\mathrm{tr}(A)$ with foci α and β.

The Davis–Wielandt shell $DW(A)$ is an ellipsoid without the interior centered at $\left(\frac{\mathrm{tr}(A)}{2}, \frac{\mathrm{tr}(A^*A)}{2}\right)$.

Assume that $W(A) = -W(A)$. Then

$$W(A + \lambda I) = W(-A + \lambda I), \quad \forall \lambda \in \mathbb{C}.$$

It follows that for every $\lambda \in \mathbb{C}$ there exists a unitary $U_\lambda \in B(H)$ such that $A + \lambda I = U_\lambda^*(-A + \lambda I)U_\lambda$, so $\|A + \lambda I\| = \|A - \lambda I\|$, that is, $A \perp_R I$.

Theorem

Let $A \in B(H)$, $\dim H = 2$. *Then the following conditions are mutually equivalent:*

(i) $A \perp_R I$,

(ii) $W(A) = -W(A)$,

(iii) $\mathrm{tr}(A) = 0$.

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