Inequalities on the spectral radius, operator norm and numerical radius of Hadamard weighted geometric mean of positive kernel operators

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Denote by $A \circ B = [a_{ij} b_{ij}]$ Hadamard (Schur) product of matrices A and B; $A^{(t)} = [a_{ij}^t]$ Hadamard (Schur) power for $t > 0$; AB denotes the usual product; $r(\cdot)$ denotes the spectral radius

X. Zhan (2009) **conjectured** that

$r(A \circ B) \leq r(AB)$ for (entrywise) non-negative $n \times n$ matrices A and B

Confirmed by K.M.R. Audenaert (2010, LAA) by proving (via **trace description** of $r(\cdot)$)

$$r(A \circ B) \leq r^{\frac{1}{2}}((A \circ A)(B \circ B)) \leq r(AB). \tag{1}$$

R.A. Horn and F. Zhang (2010, ELA) proved

$$r(A \circ B) \leq r^{\frac{1}{2}}(AB \circ BA) \leq r(AB) \tag{2}$$

(by using that Hadamard product is a **principal submatrix of the Kronecker product**)

Huang (2011, LAA): $r(A_1 \circ \cdots \circ A_m) \leq r(A_1 A_2 \cdots A_m)$ by applying Horn-Zhang technique

A.R. Schep (2011, ELA) generalized (1) and (2) to non-negative matrices that define operators on l^p, $1 \le p < \infty$. He also showed that

$$r\left(A^{(\frac{1}{2})} \circ B^{(\frac{1}{2})}\right) \le r(AB)^{\frac{1}{2}}$$

holds for a Hadamard geometric mean $A^{\left(\frac{1}{2}\right)} \circ B^{\left(\frac{1}{2}\right)}$ of **positive kernel operators** A and B on $L^p(X, \mu)$, $1 \le p < \infty$.

P. (2012, LAA) generalized Huang's result to the setting of non-negative matrices that define operators on sequence spaces; proved analogues for the **generalized** and **joint spectral radius** of bounded sets of such matrices; and **max-algebra** versions

Schep (2011) also proved

$$r(A \circ B) \le r^{\frac{1}{2}}((A \circ A)(B \circ B)) \le r^{\frac{1}{2}}(AB \circ AB) \le r(AB) \tag{3}$$

It follows trivially (P. 2012) that

$$r(A \circ B) \le r^{\frac{1}{2}}((A \circ A)(B \circ B)) \le r^{\frac{1}{2}}(BA \circ BA) \le r(AB). \tag{4}$$

It turns out that $r(AB \circ BA)$, $r(BA \circ BA)$ and $r(AB \circ AB)$ may be **different** and that $r((A \circ A)(B \circ B)) \le r(AB \circ BA)$ is **false** in general

It also holds (P. 2012) that

$$r(A \circ B) \le r^{\frac{1}{2}}((A \circ A)(B \circ B))$$

$$\le r(AB \circ AB)^{\frac{1}{4}} r(BA \circ BA)^{\frac{1}{4}} \le r(AB) \tag{5}$$

and

$$r(A \circ B) \le r^{\frac{1}{2}}(AB \circ BA)$$

$$\le r(AB \circ AB)^{\frac{1}{4}} r(BA \circ BA)^{\frac{1}{4}} \le r(AB) \tag{6}$$

for non-negative matrices that define operators on $L \in \mathcal{L}$, where $\mathcal{L}$ denotes the set of all Banach sequence spaces such that $e_n = \chi_{\{n\}} \in L$ and $\|e_n\| = 1$ for all $n \in R$ ($R = \mathbb{N}$ or $R = \{1, \ldots, N\}$)

Definition of a Banach sequence space

- $R = \mathbb{N}$ or $R = \{1, \ldots, N\}$; $S(R)$ the vector space of complex sequences $(x_i)_{i \in R}$

- $L \subseteq S(R)$ **Banach sequence space** (B.s.s.):
  - L Banach space
  - $x \in S(R)$, $y \in L$ and $|x| \leq |y| \Rightarrow x \in L$ and $\|x\|_L \leq \|y\|_L$. (**order ideal property** and **monotonicity of the norm**)

- standard examples: $l^p, 1 \leq p \leq \infty$; c_0, $l^2 \times c_0$, ...

Definition of a Banach function space

- (X, μ) σ-finite measure space; $M(X, \mu)$ a vector space of all equivalence classes of (a.e. equal) complex measurable functions on X

- $L \subseteq M(X, \mu)$ **Banach function space** (B.f.s.):
  - L Banach space
  - $f \in M(X, \mu)$, $y \in L$ and $|f| \leq |g| \Rightarrow f \in L$ and $\|f\|_L \leq \|g\|_L$.

- We assume that X is the carrier of L: there is no subset $Y \subset X$, $\mu(Y) > 0$ such that $f = 0$ a.e. on Y for all $f \in L$.

Let L a B.f.s. and L_+ a positive cone: $L_+ = \{f \in L : f \geq 0 \text{ a.e.}\}$.

An operator A on L is called a **kernel operator** if there exists a $\mu \times \mu$-measurable function $a(x, y)$ on $X \times X$ such that, for all $f \in L$ and for almost all $x \in X$,

$$\int_X |a(x, y)f(y)| \, d\mu(y) < \infty \quad \text{and} \quad (Af)(x) = \int_X a(x, y)f(y) \, d\mu(y).$$

A is positive ($AL_+ \subset L_+$) iff its kernel a is non-negative almost everywhere. We denote $A \geq 0$.

If A is positive, then it is bounded and $r(A) \in \sigma(A)$.

The above results follow from:

Theorem 1 *(Drnovšek, P. (06, Positivity); P. (09, Positivity))* L *B.f.s.,* $\{A_{ij}\}_{i=1,j=1}^{k,m} \geq 0$, $\{A_j\}_{j=1}^{m} \geq 0$ *kernel operators on* L *,* α_1, α_2,..., α_m *positive numbers with* $\sum_{i=1}^{m} \alpha_i = 1$. *Then*

$$\left(A_{11}^{(\alpha_1)} \circ \cdots \circ A_{1m}^{(\alpha_m)}\right) \cdots \left(A_{k1}^{(\alpha_1)} \circ \cdots \circ A_{km}^{(\alpha_m)}\right)$$

$$\leq (A_{11} \cdots A_{k1})^{(\alpha_1)} \circ \cdots \circ (A_{1m} \cdots A_{km})^{(\alpha_m)},$$

$$\| \left(A_{11}^{(\alpha_1)} \circ \cdots \circ A_{1m}^{(\alpha_m)}\right) \cdots \left(A_{k1}^{(\alpha_1)} \circ \cdots \circ A_{km}^{(\alpha_m)}\right) \|$$

$$\leq \|A_{11} \cdots A_{k1}\|^{\alpha_1} \cdots \|A_{1m} \cdots A_{km}\|^{\alpha_m},$$

$$r\left(\left(A_{11}^{(\alpha_1)}\circ\cdots\circ A_{1m}^{(\alpha_m)}\right)\ldots\left(A_{k1}^{(\alpha_1)}\circ\cdots\circ A_{km}^{(\alpha_m)}\right)\right)$$

$$\leq r\left(A_{11}\cdots A_{k1}\right)^{\alpha_1}\ldots r\left(A_{1m}\cdots A_{km}\right)^{\alpha_m}.$$

If, in addition, $L = L^2(X,\mu)$ and w is numerical radius, then

$$w\left(\left(A_{11}^{(\alpha_1)}\circ\cdots\circ A_{1m}^{(\alpha_m)}\right)\ldots\left(A_{k1}^{(\alpha_1)}\circ\cdots\circ A_{km}^{(\alpha_m)}\right)\right)$$

$$\leq w\left(A_{11}\cdots A_{k1}\right)^{\alpha_1}\ldots w\left(A_{1m}\cdots A_{km}\right)^{\alpha_m}.$$

In particular, we have

$$\|A_1^{(\alpha_1)} \circ A_2^{(\alpha_2)} \circ \cdots \circ A_m^{(\alpha_m)}\| \le \|A_1\|^{\alpha_1} \, \|A_2\|^{\alpha_2} \cdots \|A_m\|^{\alpha_m},$$

$$r(A_1^{(\alpha_1)} \circ \cdots \circ A_m^{(\alpha_m)}) \le r(A_1)^{\alpha_1} \cdots r(A_m)^{\alpha_m}.$$

If, in addition, $L = L^2(X, \mu)$, then

$$w(A_1^{(\alpha_1)} \circ \cdots \circ A_m^{(\alpha_m)}) \le w(A_1)^{\alpha_1} \cdots w(A_m)^{\alpha_m}.$$

If $L \in \mathcal{L}$ B.s.s., then all the above inequalities hold also for the **operator norm** and the **spectral radius** if $\sum_{i=1}^m \alpha_i \ge 1$.

The proof of Theorem 1 is based on the **sharpened version** of **Young's inequality**

$$x^{\alpha}y^{1-\alpha} = \inf_{t>0}\left\{\alpha\, t^{\frac{1}{\alpha}}x + (1-\alpha)t^{-\frac{1}{1-\alpha}}y\right\},$$

(where $x, y \geq 0, \ \ \alpha \in (0,1)$) and **Gelfand formula** for the spectral radius.

Some proofs of the above results by Theorem 1:

Proof of (3*): $r(A \circ B) \le r^{\frac{1}{2}}((A^{(2)}B^{(2)})^{(\frac{1}{2})} \circ (B^{(2)}A^{(2)})^{(\frac{1}{2})})$
$\le r^{\frac{1}{2}}((A \circ A)(B \circ B)) \le r(AB \circ AB)^{\frac{1}{4}} r(BA \circ BA)^{\frac{1}{4}} \le r(AB)$;

$(A \circ B)^2 = (A \circ B)(B \circ A) =$
$((A^{(2)})^{(\frac{1}{2})} \circ (B^{(2)})^{(\frac{1}{2})})((B^{(2)})^{(\frac{1}{2})} \circ (A^{(2)})^{(\frac{1}{2})}) \le (A^{(2)}B^{(2)})^{(\frac{1}{2})} \circ (B^{(2)}A^{(2)})^{(\frac{1}{2})}$.

It follows $r(A \circ B)^2 \le r((A^{(2)}B^{(2)})^{(\frac{1}{2})} \circ (B^{(2)}A^{(2)})^{(\frac{1}{2})})$
$\le r(A^{(2)}B^{(2)})^{\frac{1}{2}} r(B^{(2)}A^{(2)})^{\frac{1}{2}} = r(A^{(2)}B^{(2)})$.

Clearly $r(A^{(2)}B^{(2)}) = r((A \circ A)(B \circ B)) \le r(AB \circ AB) \le r(AB)^2$.

The proof of (6*): $r(A \circ B) \leq r^{\frac{1}{2}}((A^{(2)}B^{(2)})^{(\frac{1}{2})} \circ (B^{(2)}A^{(2)})^{(\frac{1}{2})})$
$\leq r^{\frac{1}{2}}(AB \circ BA) \leq r(AB \circ AB)^{\frac{1}{4}} r(BA \circ BA)^{\frac{1}{4}} \leq r(AB)$;

Similarly

$(A \circ B)^2 = (A \circ B)(B \circ A) \leq (A^{(2)}B^{(2)})^{(\frac{1}{2})} \circ (B^{(2)}A^{(2)})^{(\frac{1}{2})}$
$\leq AB \circ BA$ and thus
$r(A \circ B)^2 \leq r((A^{(2)}B^{(2)})^{(\frac{1}{2})} \circ (B^{(2)}A^{(2)})^{(\frac{1}{2})}) \leq r(AB \circ BA)$;

$r(AB \circ BA) = r((AB \circ AB)^{(\frac{1}{2})} \circ (BA \circ BA)^{(\frac{1}{2})})$
$\leq r(AB \circ AB)^{\frac{1}{2}} r(BA \circ BA)^{\frac{1}{2}} \leq r(AB) r(BA) = r(AB)^2$,
which proves (6*).

The inequality $r(A_1 \circ \cdots \circ A_m) \le r(A_1 A_2 \cdots A_m)$ and its refinements in the sense of (3)-(6) are proved in a similar way (P., 2012 ; Drnovšek, P. 2016; P. 2017+)
In particular, we have

Theorem 2 *If $L \in \mathcal{L}$, then we have for all $m \in \mathbb{N}$, $t \in [1, m]$*
(i) (Drnovšek, P. 2016)
$r(A_1 \circ \cdots \circ A_m) \le r(A_1^{(t)} \cdots A_m^{(t)})^{\frac{1}{t}}$
$\le r((A_1 \cdots A_m)^{(t)})^{\frac{1}{t}} \le r(A_1 \cdots A_m)$,
(ii) (P. 2017+)
$r(A_1 \circ \cdots \circ A_m) \le r((A_{\sigma_1(1)} \cdots A_{\sigma_1(m)}) \circ \cdots \circ (A_{\sigma_l(1)} \cdots A_{\sigma_l(m)}))^{1/l}$
$\le r(A_1 \cdots A_m)$, *where $l \le m(m-1)\ldots 1$ and $\sigma_1, \cdots, \sigma_l$ cyclic permutations of the set $\{1, \ldots, m\}$.*

If $L \in \mathcal{L}$ and A and B be non-negative matrices that define operators on L, then we have for every $t \in [1,2]$,

$$\|(A \circ B)^2\| \leq \|(A^{(t)}B^{(t)})^{(\frac{1}{t})} \circ (B^{(t)}A^{(t)})^{(\frac{1}{t})}\| \leq \left(\|A^{(t)}B^{(t)}\|\|B^{(t)}A^{(t)}\|\right)^{\frac{1}{t}}$$

$$\leq (\|(AB)^{(t)}\|\|(BA)^{(t)}\|)^{\frac{1}{t}} \leq \|AB\|\|BA\|.$$

If, in addition, $L = l^2(R)$, then

$$w((A \circ B)^2) \leq w\left((A^{(2)}B^{(2)})^{(\frac{1}{2})} \circ (B^{(2)}A^{(2)})^{(\frac{1}{2})}\right) \leq$$

$$\leq \left(w(A^{(2)}B^{(2)})\, w(B^{(2)}A^{(2)})\right)^{\frac{1}{2}} \leq \left(w((AB)^{(2)})\, w((BA)^{(2)})\right)^{\frac{1}{2}}.$$

Similarly (Drnovšek, P. 2016; P. 2017+), for a Hadamard geometric mean $A^{\left(\frac{1}{2}\right)} \circ B^{\left(\frac{1}{2}\right)}$ of positive kernel operators A and B on a Banach function space L, we have

$$\rho\left(A^{(\frac{1}{2})} \circ B^{(\frac{1}{2})}\right) \le \rho\left((AB)^{(\frac{1}{2})} \circ (BA)^{(\frac{1}{2})}\right)^{\frac{1}{2}} \le \rho(AB)^{\frac{1}{2}}.$$

More generally ,

$$\rho\left(A_1^{\left(\frac{1}{m}\right)} \circ A_2^{\left(\frac{1}{m}\right)} \circ \cdots \circ A_m^{\left(\frac{1}{m}\right)}\right)$$

$$\le \rho\left(P_1^{\left(\frac{1}{m}\right)} \circ P_2^{\left(\frac{1}{m}\right)} \circ \cdots \circ P_m^{\left(\frac{1}{m}\right)}\right)^{\frac{1}{m}} \le \rho(A_1 A_2 \cdots A_m)^{\frac{1}{m}},$$

where $P_j = A_j \ldots A_m A_1 \ldots A_{j-1}$ for $j = 1, \ldots, m$.

Also

$$\left\| \left(A_1^{\left(\frac{1}{m}\right)} \circ A_2^{\left(\frac{1}{m}\right)} \circ \cdots \circ A_m^{\left(\frac{1}{m}\right)} \right)^m \right\|$$

$$\leq \left\| P_1^{(\frac{1}{m})} \circ P_2^{(\frac{1}{m})} \circ \cdots \circ P_m^{(\frac{1}{m})} \right\| \leq \|P_1\|^{\frac{1}{m}} \|P_2\|^{\frac{1}{m}} \cdots \|P_m\|^{\frac{1}{m}}.$$

If, in addition, $L = L^2(X, \mu)$, then

$$w\left(\left(A_1^{\left(\frac{1}{m}\right)} \circ A_2^{\left(\frac{1}{m}\right)} \circ \cdots \circ A_m^{\left(\frac{1}{m}\right)} \right)^m \right)$$

$$\leq w\left(P_1^{(\frac{1}{m})} \circ P_2^{(\frac{1}{m})} \circ \cdots \circ P_m^{(\frac{1}{m})} \right) \leq w(P_1)^{\frac{1}{m}} w(P_2)^{\frac{1}{m}} \cdots w(P_m)^{\frac{1}{m}}.$$

In the special case $L = L^2(X, \mu)$ we also prove by using $\|A\|^2 = \rho(AA^*) = \rho(A^*A)$ that

$$\|A^{(\frac{1}{2})} \circ B^{(\frac{1}{2})}\| \leq \rho\left((A^*B)^{(\frac{1}{2})} \circ (B^*A)^{(\frac{1}{2})}\right)^{\frac{1}{2}} \leq \rho(A^*B)^{\frac{1}{2}}$$

$$(\leq \|AB^*\|^{\frac{1}{2}} \leq \|A\|^{\frac{1}{2}}\|B\|^{\frac{1}{2}}).$$

If, in addition, $L = l^2(R)$ and $\alpha \geq \frac{1}{2}$, then

$$\|A^{(\alpha)} \circ B^{(\alpha)}\| \leq \rho^{\frac{1}{2}}\left((A^TB)^{(\alpha)} \circ (B^TA)^{(\alpha)}\right) \leq \rho^{\alpha}(A^TB),$$

whenever $\alpha \geq \frac{1}{2}$.

Theorem 3 *Let A and B be positive kernel operators on $L^2(X, \mu)$. Then*

$$\|A^{\left(\frac{1}{3}\right)} \circ (B^*)^{\left(\frac{1}{3}\right)} \circ A^{\left(\frac{1}{3}\right)}\| \leq$$

$$\rho\left((A^*B^*A^*ABA)^{\left(\frac{1}{3}\right)} \circ (BAA^*B^*A^*A)^{\left(\frac{1}{3}\right)} \circ (A^*ABAA^*B^*)^{\left(\frac{1}{3}\right)}\right)^{\frac{1}{6}} \leq \|ABA\|^{\frac{1}{3}}.$$

If, in addition, $L = l^2(R)$ and $\alpha \geq \frac{1}{3}$, then

$$\|A^{(\alpha)} \circ (B^T)^{(\alpha)} \circ A^{(\alpha)}\| \leq$$

$$\rho((A^TB^TA^TABA)^{(\alpha)} \circ (BAA^TB^TA^TA)^{(\alpha)} \circ (A^TABAA^TB^T)^{(\alpha)})^{\frac{1}{6}} \leq \|ABA\|^{\alpha}.$$

Several of the above results can be:

- **refined**;
- generalized to the setting of **generalized** and **joint** spectral radius of bounded sets of positive kernel operators and-or matrices;
- **max-algebra** versions of the above results also hold.

1. P., Inequalities on the spectral radius, operator norm and numerical radius of Hadamard weighted geometric mean of positive kernel operators, (2016), arxiv.org/abs/1612.01767

2. P., Bounds on the joint and generalized spectral radius of Hadamard ..., (2016), arxiv.org/abs/1612.01765

3. Drnovšek and P., Inequalities on the spectral radius and the operator norm ... on sequence spaces, **Banach J. Math. Anal.** (2016).

4. P. , Bounds on the generalized and the joint spectral radius ... on sequence spaces, **Linear Algebra Appl.** (2012).

5. P., Inequalities for the spectral radius of non-negative functions, **Positivity** (2009).

6. Drnovšek and P., Inequalities for the Hadamard weighted geometric mean ... , **Positivity** (2006).

Generalized and joint spectral radius

Let Σ be a bounded set of bounded operators on L. For $m \geq 1$, let $\Sigma^m = \{A_1 A_2 \cdots A_m : A_i \in \Sigma\}$.

The generalized spectral radius of Σ is defined by

$$r(\Sigma) = \limsup_{m\to\infty} [\sup_{A\in\Sigma^m} r(A)]^{1/m}.$$

The joint spectral radius of Σ is defined by

$$\widehat{r}(\Sigma) = \lim_{m\to\infty}[\sup_{A\in\Sigma^m} \|A\|]^{1/m}.$$

It is well known that $r(\Sigma) = \widehat{r}(\Sigma)$ for a precompact set Σ of compact operators on L (Berger-Wang; Schulman-Turovskii), in particular for a bounded set of complex $n \times n$ matrices. In general $r(\Sigma)$ and $\widehat{r}(\Sigma)$ may differ.

Let Ψ and Σ be sets of non-negative matrices that define operators on L and $\alpha > 0$. Then $\Psi \circ \Sigma$ and $\Psi^{(\alpha)}$ denote respectively the *Hadamard (Schur) product* of Ψ and Σ and the *Hadamard (Schur) power* of Ψ, e.g.,

$\Psi \circ \Sigma = \{A \circ B : A \in \Psi, B \in \Sigma\}$ and

$\Psi^{(\alpha)} = \{A^{(\alpha)} : A \in \Psi\}$.