On matrix theory, graph theory, and finite geometry

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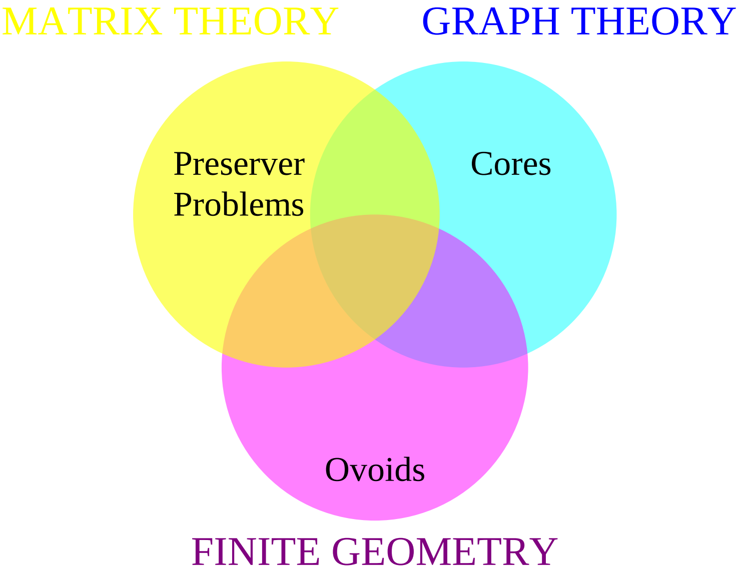
LAW 2017, Ljubljana, June 16, 2017


MATRIX THEORY
GRAPH THEORY
Preserver
Problems
Cores
Ovoids
FINITE GEOMETRY

Preserver Problems

Preserver problems in matrix theory

A typical PRESERVER PROBLEM demands a characterization of all maps

$$\Phi : \mathcal{M} \to \mathcal{M}$$

on a set $\mathcal{M}$ of matrices that preserve some

- subset
- function
- relation
- etc.

Sometimes there are some additional assumptions on Φ: linearity, additivity, bijectivity, etc.

Preserver problems

Example 1: Determinant preservers (Frobenius, 1897)

A linear bijective map $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ preserves determinant, that is, $\det \Phi(A) = \det A$ for all A, if and only if

$$\Phi(A) = PAQ \quad \text{or} \quad \Phi(A) = PA^\top Q,$$

where $\det(PQ) = 1$.

Example 2: Invertibility preservers

Let $\mathbb{F}$ be a field with $|\mathbb{F}| \geq 3$. A linear bijective map $\Phi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ preserves invertibility in both directions, that is, $A \in GL_n(\mathbb{F}) \iff \Phi(A) \in GL_n(\mathbb{F})$ if and only if

$$\Phi(A) = PAQ \quad \text{or} \quad \Phi(A) = PA^\top Q,$$

where $P, Q \in GL_n(\mathbb{F})$.

Preserver problems

Example 3: Adjacency preservers

A BIJECTIVE map $\Phi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ preserves adjacency in BOTH directions, that is, $\mathrm{rk}(A - B) = 1 \iff \mathrm{rk}(\Phi(A) - \Phi(B)) = 1$ iff

$$\Phi(A) = PA^{\sigma}Q + B \quad \text{or} \quad \Phi(A) = P(A^{\sigma})^{\top}Q + B,$$

where $P, Q \in GL_n(\mathbb{F})$, $B \in M_n(\mathbb{F})$, and field automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ is applied entry-wise.

The above result is called the *fundamental theorem of geometry of matrices* (for $M_n(\mathbb{F})$). Example 2 follows from it by observing:

$$\mathrm{rk}\, A = 1$$

$$\iff$$

$$\forall B \in GL_n(\mathbb{F}) : \; B + \lambda A \in GL_n(\mathbb{F}) \text{ for all but at most one } \lambda \in \mathbb{F}$$

Fundamental theorems of geometry of matrices $(m, n \geq 2)$

$M_{m\times n}(\mathbb{F})$ = rectangular matrices

$$\Phi(A) = PA^\sigma Q + B \quad \text{or} \quad \Phi(A) = P(A^\sigma)^\top Q + B \quad (m = n)$$

$S_n(\mathbb{F})$ = symmetric matrices

$$\text{If } (\mathbb{F}, n) \neq (\mathbb{F}_2, 3): \qquad \Phi(A) = aPA^\sigma P^\top + B$$

$H_n(\mathbb{F})$ = hermitian matrices

$$\Phi(A) = aPA^\sigma P^* + B$$

$A_n(\mathbb{F})$ = alternate matrices

$$\text{If } n \geq 5: \qquad \Phi(A) = aPA^\sigma P^\top + B$$

Adjacency preservers (one direction, no bijectivity)

- $H_n(\mathbb{C})$ (Šemrl, Huang 2008, Canad. J. Math.)
- $H_2(\mathbb{D})$ (Huang 2008, Aequationes Math.)
- $S_n(\mathbb{R})$ (Legiša 2011, Math. Commun.)
- $H_n(\mathbb{F}_{q^2})$ (Orel 2009, Finite Fields Appl.)
- $S_n(\mathbb{F}_q)$ (Orel 2012, J. Algebraic Combin.)
- $M_{m\times n}(\mathbb{D})$ (Šemrl 2014, Mem. Amer. Math. Soc.)
  (de Seguins Pazzis & Šemrl, 2015, J. Algebra)
  (Huang & Šemrl 2016, Linear Algebra Appl.)
- $M_{m\times n}(\mathbb{F}_q)$(Huang, Huang, Li, Sze 2014, Linear Algebra Appl.)
- $HGL_n(\mathbb{F}_{q^2})$, $q \geq 4$ (Orel 2016, Linear Algebra Appl.)
- $A_n(\mathbb{F}_q)$ (Huang, Huang, Zhao 2015, Discrete Math.)

Graph theory

Graphs: undirected (finite) without loops/multiple edges

A *homomorphism* between graphs Γ_1 and Γ_2 is a map $\Phi : V(\Gamma_1) \to V(\Gamma_2)$ such that

$$\{u, v\} \in E(\Gamma_1) \Longrightarrow \{\Phi(u), \Phi(v)\} \in E(\Gamma_2).$$

An *isomorphism* between graphs Γ_1 and Γ_2 is a bijective map $\Phi : V(\Gamma_1) \to V(\Gamma_2)$ such that

$$\{u, v\} \in E(\Gamma_1) \Longleftrightarrow \{\Phi(u), \Phi(v)\} \in E(\Gamma_2).$$

If $\Gamma_1 = \Gamma_2$, then
homomorphism = *endomorphism*
isomorphism = *automorphism*.

For finite graphs: bijective endomorphism = automorphism

Example

The Fundamental theorem of geometry of matrices in $M_n(\mathbb{F})$ characterizes all automorphisms of Γ with

$$V(\Gamma) := M_n(\mathbb{F}) \qquad E(\Gamma) := \big\{\{A, B\} \ : \mathrm{rk}(A - B) = 1\big\},$$

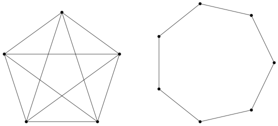
Adjacency preservers (in 1 direction) are the endomorphisms of Γ.

Cores

A graph is a *core* if all its endomorphisms are automorphisms.

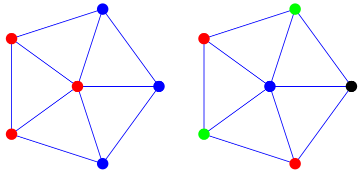
Basic examples:

- complete graphs
- odd cycles



The *clique number* $\omega(\Gamma)$ of graph Γ is the largest number of pairwise adjacent vertices.

The *chromatic number* $\chi(\Gamma)$ of graph Γ is the smallest number of colors needed to color the vertices in such way that adjacent vertices get different colors.



$\omega(\Gamma) = 3$ $\qquad$ $\chi(\Gamma) = 4$

$$\chi(\Gamma) \geq \omega(\Gamma)$$

Many graphs are either cores or $\chi(\Gamma) = \omega(\Gamma)$.

Cameron, Kazanidis 2008, J. Aust. Math. Soc.

If $\mathrm{Aut}(\Gamma)$ acts transitively on unordered pairs of non-adjacent vertices, then Γ is a core or $\chi(\Gamma) = \omega(\Gamma)$.

Godsil, Royle 2011, Ann. Comb.

If Γ is connected regular and $\mathrm{Aut}(\Gamma)$ acts transitively on unordered pairs of vertices at distance 2, then Γ is a core or $\chi(\Gamma) = \omega(\Gamma)$.

To show that $\chi(\Gamma) > \omega(\Gamma)$ for a particular graph, lower bounds for $\chi(\Gamma)$ can be useful:

Hoffman, 1970

If $E(\Gamma) \neq \emptyset$, then

$$\chi(\Gamma) \geq 1 + \frac{\lambda_{\mathsf{max}}}{-\lambda_{\mathsf{min}}}.$$

Example 1: Graph from Orel 2009, Finite Fields Appl.:

$$\begin{aligned} V(\Gamma) &= H_n(\mathbb{F}_{q^2}) \\ E(\Gamma) &= \big\{\{A, B\} : \mathrm{rk}(A - B) = 1\big\} \end{aligned}$$

Eigenvalues: $\frac{(-q)^{2n-r}-1}{q+1} \qquad (r = 0, 1, \ldots, n).$

$$\chi(\Gamma) \geq 1 + \frac{\lambda_{\max}}{-\lambda_{\min}} > q = \omega(\Gamma).$$

Godsil & Royle $\Longrightarrow$ Γ is a core $\Longrightarrow$ Adjacency preservers:

$$\Phi(A) = PA^{\sigma}P^{*} + B$$

Example 2: Finite graph from

- Šemrl 2014, Mem. Amer. Math. Soc.
- Huang, Huang, Li, Sze 2014, Linear Algebra Appl.

$$V(\Gamma) = M_{m\times n}(\mathbb{F}_q)$$
$$E(\Gamma) = \{\{A, B\} : \mathrm{rk}(A - B) = 1\}$$

Eigenvalues: $\frac{q^{m+n-r}-q^m-q^n+1}{q-1}$ $(r = 0, 1, \ldots, \min\{m, n\})$.

$$\chi(\Gamma) \geq 1 + \frac{\lambda_{\max}}{-\lambda_{\min}} = q^{\max\{m,n\}} = \omega(\Gamma).$$

Roberson 2016, arXiv

Every endomorphism of a primitive strongly regular graph is either an automorphism or a coloring.

Adjacency preservers on $M_{n\times 2}(\mathbb{F}_q)$ or $M_{2\times n}(\mathbb{F}_q)$:

$$\Phi(A) = PA^\sigma Q + B$$
$$\Phi(A) = P(A^\sigma)^\top Q + B \quad (n = 2)$$
$$\mathrm{Image}(\Phi) = \{\text{pairwise adjacenct matrices}\}$$

Finite geometry

Let $q = p^k$, p an *odd* prime, and $A = A^\top \in GL_n(\mathbb{F}_q)$. The set

$$\mathcal{Q} = \{\langle \mathbf{x} \rangle : \mathbf{x}^\top A \mathbf{x} = 0, \mathbf{x} \neq 0\}$$

is a *quadric*.
Here, $\langle \mathbf{x} \rangle$=1-dimensional subspace that is spanned by $\mathbf{x} \in \mathbb{F}_q^n$.

Definition

If n is odd, then the quadric is *parabolic*.
If n is even, then the quadric is *hyperbolic* if $|\mathcal{Q}| = \frac{(q^{n/2}-1)(q^{n/2-1}+1)}{q-1}$, and *elliptic* if $|\mathcal{Q}| = \frac{(q^{n/2}+1)(q^{n/2-1}-1)}{q-1}$.

A projective subspace V is *totally singular* if $\mathbf{x}^\top A \mathbf{y} = 0$ for all points $\langle \mathbf{x} \rangle$, $\langle \mathbf{y} \rangle$ in V.
Maximal totally singular subspaces are *generators* and all of them have the same dimension.

Ovoid - definition

An *ovoid* of an orthogonal polar space is a set of points meeting every generator in precisely one point.

Spread - definition

A *spread* of an orthogonal polar space is a set of generators that partition the point set $\mathcal{Q}$.

Existence of ovoids/spreads is a long standing open problem.

The *point graph of an orthogonal polar space* is graph $Q^{\varepsilon}_{n-1}(q)$ s.t.

$$V\big(Q^{\varepsilon}_{n-1}(q)\big) = \mathcal{Q},$$
$$E\big(Q^{\varepsilon}_{n-1}(q)\big) = \big\{\{\langle \mathbf{x}\rangle, \langle \mathbf{y}\rangle\} : \mathbf{x}^\top A\mathbf{y} = 0, \langle \mathbf{x}\rangle \neq \langle \mathbf{y}\rangle\big\}.$$

Cameron, Kazanidis 2008, J. Aust. Math. Soc.

- $Q^{\varepsilon}_{n-1}(q)$ is a core $\iff$ the associated polar space does not have a partition into ovoids.
- The complement $\overline{Q^{\varepsilon}_{n-1}(q)}$ is a core $\iff$ the associated polar space has not an ovoid or has not a spread.

Analogous result on a unitary polar space was applied in:

Orel, LAA 2016

Adjacency preservers $\Phi : HGL_n(\mathbb{F}_{q^2}) \to HGL_n(\mathbb{F}_{q^2})$ for $q \geq 4$ are:

$$\Phi(A) = PA^{\sigma}P^{*} \qquad \Phi(A) = P(A^{-1})^{\sigma}P^{*}$$

Huang, Huang, Zhao; Discrete Math. 2015

Let q be odd. Adjacency preservers $\Phi : A_4(\mathbb{F}_q) \to A_4(\mathbb{F}_q)$ with

$$\mathrm{Image}(\Phi) = \{\text{pairwise adjacenct matrices}\}$$

exist iff $Q^{+}_{7}(q)$ has a spread.

Let q be odd, $A = A^\top \in GL_n(\mathbb{F}_q)$. Lester (Canad. J. Math. 1977) characterized bijective maps $\Phi : \mathbb{F}_q^n \to \mathbb{F}_q^n$ that satisfy:

$$(\mathbf{x}-\mathbf{y})^\top A(\mathbf{x}-\mathbf{y}) = 0,\ \mathbf{x} \neq \mathbf{y}$$
$$\Longrightarrow$$
$$\big(\Phi(\mathbf{x})-\Phi(\mathbf{y})\big)^\top A\big(\Phi(\mathbf{x})-\Phi(\mathbf{y})\big) = 0,\ \Phi(\mathbf{x}) \neq \Phi(\mathbf{y})$$

i.e., the automorphisms of the *Affine polar graph* $VO_n^\varepsilon(q)$:

$$V\big(VO_n^\varepsilon(q)\big) = \mathbb{F}_q^n$$
$$E\big(VO_n^\varepsilon(q)\big) = \big\{\{\mathbf{x},\mathbf{y}\} : (\mathbf{x}-\mathbf{y})^\top A(\mathbf{x}-\mathbf{y}) = 0,\ \mathbf{x} \neq \mathbf{y}\big\}$$

If $A = M = diag(1,-1,\dots,-1)$, $-1 \in \mathbb{F}_q$ is not a square, $n \geq 3$:

$$\Phi(\mathbf{x}) = \alpha L\mathbf{x}^\sigma + \mathbf{x}_0$$
$$\Phi(\mathbf{x}) = \alpha K\mathbf{x}^\sigma + \mathbf{x}_0 \qquad (n \text{ is even})$$

$L^\top ML = M$, $K^\top MK = -M$, $\sigma : \mathbb{F}_q \to \mathbb{F}_q$ a field automorphism.

What if there is no bijectivity assumption?

$VO_n^+(q)$	$VO_n^-(q)$	$VO_n(q)$
n even, A hyperbolic	n even, A elliptic	n odd, A parabolic

Proposition (Orel, J. Combin. Theory Ser. A, 2017)

Either $\Gamma = VO_n^\varepsilon(q)$ is a core or $\chi(\Gamma) = \omega(\Gamma)$.

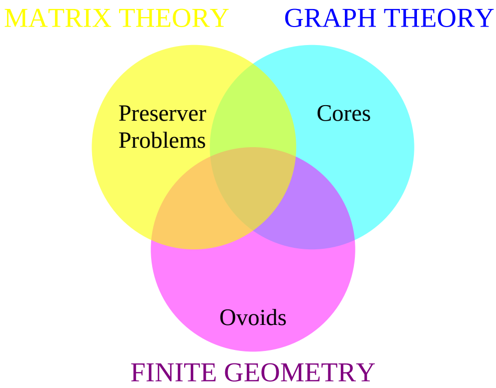
Theorem (Orel, JCTA 2017)

If $n \geq 4$ and q is odd, then $VO_n^-(q)$ is a core.

Theorem (Orel, JCTA 2017)

Let $\Gamma = VO_n^\varepsilon(q)$ be parabolic or hyperbolic (with Witt index ≥ 2).
If $\chi(\Gamma) = \omega(\Gamma)$, then $Q_{n-1}^\varepsilon(q)$ has an ovoid.
(A 'weak' backward implication is also true.)

Constructions of known ovoids can be used to construct (weird) nonbijective maps (endomorphisms) Φ.



A survey:

M. Orel, *Preserver Problems over Finite Fields.* In: J. Simmons (Ed.), Finite Fields: Theory, Fundamental Properties and Applications, Nova Science Publishers, New York, 2017, pp. 1–54.

Thank you for your attention!