Circles in the spectrum and numerical ranges

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$$W_e(\mathcal{T}) = \{\lambda \in \mathbb{C}^n : \text{there exists an orthonomal sequence}$$

$$(x_k) \subset H \text{ such that } \lambda = \lim_{k \to \infty} \langle \mathcal{T} x_k, x_k\rangle\}$$

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$$\operatorname{conv} \sigma(T_1, \dots, T_n) \subset \overline{W(T_1, \dots, T_n)}$$

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Let $\mathcal{T} = (T_1, \dots, T_n) \in B(H)^n$*. Let* $\mu \in \operatorname{Int}(W_e(\mathcal{T}))$*. Then there exists an infinite-dimensional subspace* $L \subset H$

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Equivalently, $\operatorname{Int}(W_e(\mathcal{T})) \subset W_\infty(\mathcal{T})$.

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Let $\mathcal{T} \in B(H)^n$. Then $W_\infty(\mathcal{T})$ is the set of all $\lambda \in \mathbb{C}^n$ for which there exists an infinite-dimensional subspace $L \subset H$ such that

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$$W_e(\mathcal{T}) = \overline{W_\infty(\mathcal{T} + \mathcal{K})}$$

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$$(\lambda, \lambda^2, \dots, \lambda^n) \in \operatorname{Int} (W_e(T, T^2, \dots, T^n)).$$

for all $n \in \mathbb{N}$.

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THANK YOU FOR YOUR ATTENTION