# Jordan triple product homomorphisms on triangular matrices to and from dimension one

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LAW'17

# Preliminaries

**Definition**

A mapping $\Phi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{F})$ is a **Jordan triple product homomorphism (J.T.P.)** iff $\Phi(ABA) = \Phi(A)\Phi(B)\Phi(A)$ for every $A, B \in \mathcal{M}_n(\mathbb{F})$.

Easy to see: if $\operatorname{char} \mathbb{F} \neq 2$ and $\Phi$ additive, then J.T.P. property is equivalent to the definition of Jordan homomorphism:
$\Phi(ab + ba) = \Phi(a)\Phi(b) + \Phi(b)\Phi(a)$.

# Background

Motivation:

- Lu, Jordan triple maps (2003): bijective J.T.P. homomorphisms $\Phi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_n(\mathbb{F})$ are additive
- Šemrl, Maps on matrix and operator algebras (2006):
  - asked for characterization of not necessarily bijective J.T.P. homomorphisms
- Kuzma, Jordan triple product homomorphisms (2006):
  - characterized nondegenerate J.T.P. homomorphisms $\Phi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_n(\mathbb{F})$

# Background

- Dobovišek, Maps from $\mathcal{M}_n(\mathbb{F}) \to \mathbb{F}$ that are multiplicative with respect to the Jordan triple product (2008):
  - characterized J.T.P. homomorphisms $\Phi : \mathcal{M}_n(\mathbb{F}) \to \mathbb{F}$ : $\Phi(A) = \pm\varphi(\det A)$, $\varphi$ multiplicative
- Dobovišek, Maps from $\mathcal{M}_2(\mathbb{F})$ to $\mathcal{M}_3(\mathbb{F})$ that are multiplicative with respect to the Jordan triple product (2013)
- Kokol Bukovšek, Mojškerc, Jordan triple product homomorphisms on Hermitian matrices to and from dimension one (2016)
- Kokol Bukovšek, Mojškerc, Jordan triple product homomorphisms on Hermitian matrices of dimension two (2017)

# From $\mathcal{T}_n(\mathbb{C}) \to \mathbb{C}$

- Define functions $\varphi_i : \mathbb{C} \to \mathbb{C}$ as $\varphi_i(a) = \Phi(\mathrm{diag}(1, \ldots, 1, a, 1, \ldots, 1))$ with $a$ at $i$-th position.
- Define also $\sqrt{a} := \sqrt{|a|}(\cos\frac{\alpha}{2} + i \sin\frac{\alpha}{2})$ for every $a \in \mathbb{C}$ with $a = |a|(\cos\alpha + i\sin\alpha)$, $\alpha \in [0, 2\pi)$.
- Lemma: $\varphi_i$ are multiplicative unital maps.

**Corrolary**

Take arbitrary $a_1, \ldots, a_n \in \mathbb{C}$. Then

$$\Phi(\mathrm{diag}(a_1, \ldots, a_n)) = \varphi_1(a_1) \cdot \ldots \cdot \varphi_n(a_n).$$

# From $\mathcal{T}_n(\mathbb{C}) \to \mathbb{C}$

Define $E_{ij} \in \mathcal{T}_n(\mathbb{C})$ entry-wise with $e_{ij} = \begin{cases} 1; & i = k, j = l \\ 0; & \text{otherwise} \end{cases}$ for $k, l = 1, \ldots, n$ and $i \leq j$.

Crucial lemma:

**Lemma**

Take $A \in \mathcal{T}_n(\mathbb{C})$, such that $A = \text{diag}(a_{11}, \ldots, a_{nn}) + a_{ij}E_{ij}$, $a_{ii}, a_{jj} \neq 0$. Then $\Phi(A) = \varphi_1(a_{11}) \cdot \ldots \cdot \varphi_n(a_{nn})$.

Proof: Define $D = \text{diag}(d_{11}, \ldots, d_{nn})$, such that

$$d_{kk} = \begin{cases} -\frac{a_{ii}}{a_{jj}}; & j = k \\ 1; & \text{otherwise} \end{cases}.$$

Then $ADA$ is diagonal and $\Phi(ADA) = \Phi(A)$.

# From $\mathcal{T}_n(\mathbb{C}) \to \mathbb{C}$

Main theorem of this section:

**Theorem**

Take an arbitrary $A \in \mathcal{T}_n(\mathbb{C})$. Then

$$\Phi(A) = \Phi(\mathrm{diag}(a_{11}, \ldots, a_{nn})) = \varphi_1(a_{11}) \cdot \ldots \cdot \varphi_n(a_{nn}),$$

where $\varphi_i$ are unital multiplicative maps.

# From $\mathcal{T}_n(\mathbb{C}) \to \mathbb{C}$

Idea of proof: We devise an algorithm that will "diagonalise" $A$ or some power of $A$, whilst preserving the value of $\Phi(A)$.

- Choose a non-diagonal entry of $A$, namely $a_{ij}$. If $a_{ij} = 0$, there is nothing to do.
- If $a_{ij} \neq 0$ and $a_{ii} + a_{jj} \neq 0$. Define

$$B_{ij} = I - \frac{a_{ij}}{a_{ii} + a_{jj}} E_{ij}.$$

- If for $a_{ij} \neq 0$ it holds that $a_{ii} + a_{jj} = 0$ and $a_{ii} \neq 0$, define $C_{ij}$ as follows:

$$C_{ij} = I - 2E_{jj} + \frac{a_{ij}}{2a_{ii}} E_{ij}.$$

# From $\mathcal{T}_n(\mathbb{C}) \to \mathbb{C}$

Denote $F_{ij} = \begin{cases} I; & a_{ij} = 0 \quad \text{or} \quad a_{ii} = a_{jj} = 0 \\ B_{ij}; & a_{ij} \neq 0,\ a_{ii} + a_{jj} \neq 0 \\ C_{ij}; & a_{ij} \neq 0,\ a_{ii} + a_{jj} = 0 \end{cases}.$

The algorithm:

- Start with the first entry in the first diagonal above the main diagonal, $a_{12}$.
- Apply $F_{12}$ to get: $A^{(1)} := F_{12} A F_{12}$.
- Proceed to the next entry in the same diagonal, $a_{23}^{(1)}$.
- Apply $F_{23}^{(1)}$ to $A^{(1)}$ to get: $A^{(2)} := F_{23}^{(1)} A^{(1)} F_{23}^{(1)}$.
- Repeat the process. When the end of diagonal, $a_{n-1,n}^{(n-1)}$, is reached, proceed to the first entry in the next diagonal, $a_{13}^{(n-1)}$.
- Follow the process down the second diagonal.
- Repeat the process for every diagonal.

The final matrix, $A^{(k)}$, has the same diagonal entries as $A$, the only possible non-zero off-diagonal entries $a_{ij}^{(k)}$ are those with $a_{ii} = a_{jj} = 0$, and $\Phi(A^{(k)}) = \Phi(A)$.

Take $r \in \mathbb{N}$ such that $(A^{(k)})^r$ is diagonal.
Then

$$\Phi(A)^r = \Phi(A^r) = \Phi((A^{(k)})^r) = \prod_{i=1}^{n} \varphi_i(a_{ii})^r.$$

# From $\mathbb{C} \to \mathcal{T}_n(\mathbb{C})$

Lemma 1 Take $\Phi : \mathbb{F} \to \mathcal{M}_n(\mathbb{C})$ J.T.P. homomorphism. Then

$$\Phi(\lambda) = S \begin{bmatrix} \Psi(\lambda) & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & -I \end{bmatrix} S^{-1},$$

where $\Psi$ is a J.T.P. homomorphism with $\Psi(0) = 0$.

Lemma 2 Take $\Phi : \mathbb{F} \to \mathcal{M}_n(\mathbb{C})$ J.T.P. homomorphism with $\Phi(0) = 0$. Then

$$\Phi(\lambda) = S \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Psi_1(\lambda) & 0 \\ 0 & 0 & -\Psi_2(\lambda) \end{bmatrix} S^{-1}$$

for some $S \in \mathcal{M}_n(\mathbb{C})$ invertible and $\Psi_1$, $\Psi_2$ J.T.P. homomorphisms with $\Psi_1(1), \Psi_2(1) = I$.

# From $\mathbb{C} \to \mathcal{T}_n(\mathbb{C})$

**Lemma 3** Take $\Phi : \mathbb{F} \to \mathcal{M}_n(\mathbb{C})$ J.T.P. homomorphism with $\Phi(0) = 0$ and $\Phi(1) = I$. Then there exist $S \in \mathcal{M}_n(\mathbb{C})$ invertible and $\Psi_1, \Psi_2$ J.T.P. homomorphisms such that $\Psi_1(-1) = I$, $\Psi_2(-1) = -I$ and

$$\Phi(\lambda) = S \begin{bmatrix} \Psi_1(\lambda) & 0 \\ 0 & \Psi_2(\lambda) \end{bmatrix} S^{-1}.$$

# From $\mathbb{C} \to \mathcal{T}_n(\mathbb{C})$

**Special case: $\Phi$ continuous**

If a J.T.P. homomorphism $\Phi : \mathbb{R} \to \mathcal{T}_n(\mathbb{R})$ with $\Phi(0) = 0$, $\Phi(1) = I$ and $\Phi(-1) = \pm I$, is continuous, then $\Phi$ is multiplicative. Its image is uniquely determined by $\Phi(a_0)$ for some $a_0 > 0$. Let $b > 0$ and $x = \log_{a_0} b$. Then $\Phi(b) = \Phi(a_0)^x$. If $b < 0$, then $\Phi(b) = \pm\Phi(|b|)$.

# From $\mathbb{C} \to \mathcal{T}_n(\mathbb{C})$

## General structural theorem on $\mathbb{R}$

Take $\Phi : \mathbb{R} \to \mathcal{T}_n(\mathbb{F})$ a J.T.P. homomorphism with $\Phi(0) = 0$, $\Phi(1) = I$ and $\Phi(-1) = \pm I$, $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Then

$$\Phi(a) = S \begin{bmatrix} \Phi_1(a) & & 0 \\ & \ddots & \\ 0 & & \Phi_k(a) \end{bmatrix} S^{-1}$$

with

$$\Phi_i(a) = \begin{bmatrix} \varphi_i(a) & & \bigstar \\ & \ddots & \\ 0 & & \varphi_i(a) \end{bmatrix},$$

where $\varphi_i : \mathbb{R} \to \mathbb{R}$ are multiplicative maps and $S \in \mathcal{M}_n(\mathbb{F})$ invertible.

# From $\mathbb{C} \to \mathcal{T}_n(\mathbb{C})$

Special case: Let $\Phi : \mathbb{R} \to \mathcal{T}_n(\mathbb{R})$ a J.T.P. homomorphism of the form

$$\Phi(a) = \begin{bmatrix} \varphi(a) & & \star \\ & \ddots & \\ 0 & & \varphi(a) \end{bmatrix}$$

with $\varphi : \mathbb{R} \to \mathbb{R}$ multiplicative. Without the loss of generality we may assume that

$$\Phi(a_0) = S\varphi(a_0) \begin{bmatrix} 1 & \delta_1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & \delta_{n-1} \\ 0 & & & 1 \end{bmatrix} S^{-1}$$

with $\delta_i \in \{0, 1\}$,

# From $\mathbb{C} \to \mathcal{T}_n(\mathbb{C})$

and

$$\Phi(b) = S\varphi(b) \begin{bmatrix} 1 & & \psi_{ij}(x) \\ & \ddots & \\ 0 & & 1 \end{bmatrix} S^{-1},$$

where $\psi_{i,i+1} \in \{0,1\}$, $\psi_{i,i+k}(0) = 0$ for $k \geq 1$, $\psi_{i,i+k}(1) = 0$ for $k \geq 2$, $S \in \mathcal{T}_n(\mathbb{R})$ invertible, and $x = \log_{a_0} b$. Then

$$\begin{aligned}\psi_{ij}(2x+y) =& 2\psi_{ij}(x) + \psi_{ij}(y) + \sum_{k=i+1}^{j-1} (\psi_{ik}(x)\psi_{kj}(y) + \psi_{ik}(y)\psi_{kj}(x)) + \\ &+ \sum_{k=i+1}^{j-2} \sum_{s=k+1}^{j-1} \psi_{ik}(x)\psi_{ks}(y)\psi_{sj}(x)\end{aligned}$$

for every $x, y \in \mathbb{R}$.

# From $\mathbb{C} \to \mathcal{T}_n(\mathbb{C})$

## Examples

Consider case $n = 3$ of previous proposition. Then $\psi_{12}$ and $\psi_{23}$ are (not necessarily equal) additive maps. For $\psi_{13}$ it holds that

$$\psi_{13}(2x + y) = 2\psi_{12}(x) + \psi_{13}(y) + \psi_{12}(x)\psi_{23}(y) + \psi_{12}(y)\psi_{23}(x).$$

Suppose there exists $a_0$ such that $\operatorname{rank}(\Phi(a_0) - \varphi(a_0)I) = 1$ and there **doesn't** exist $a_0$ such that $\operatorname{rank}(\Phi(a_0) - \varphi(a_0)I) = 2$. Then if $\psi_{ij}$ is nonzero, it must be additive for any combination of $i, j$.

# From $\mathbb{C} \to \mathcal{T}_n(\mathbb{C})$

## General structural theorem on $\mathbb{C}$

Let $\Phi : \mathbb{C} \to \mathcal{T}_n(\mathbb{C})$ a J.T.P. homomorphism with $\Phi(0) = 0$ and $\Phi(1) = I$. Then $\Phi$ takes the form

$$\Phi(z) = S \begin{bmatrix} \Phi_1(z) & & \\ & \ddots & \\ & & \Phi_k(z) \end{bmatrix} S^{-1},$$

where

$$\Phi_i(z) = \begin{bmatrix} \varphi_i(z) & & \bigstar \\ & \ddots & \\ 0 & & \varphi_i(z) \end{bmatrix}$$

with $\varphi_i : \mathbb{C} \to \mathbb{C}$ multiplicative maps and $S \in \mathcal{M}_n(\mathbb{C})$ invertible.

Thank you for your attention!