Length realizability problem for pairs of quasi-commuting matrices

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Main definitions

Let $\mathbb{F}$ be an arbitrary field and $n > 1$ be a fixed integer. Let $S = \{A_1, \ldots, A_s\}$ be a set of $n \times n$ matrices over $\mathbb{F}$. Let us denote

- $S^0 = \{I_n\}$;
- $S^m = \{A_{i_1} \cdots A_{i_t} | 0 \leq t \leq m\}$, i.e. the set of products of length not greater than $m$ in $A_i$;
- $\mathcal{L}_m(S) = \langle S^m \rangle$;
- $\mathcal{L}(S)$ — the linear span of all products in elements from $S$.

$L_0(S) \subseteq L_1(S) \subseteq \ldots \subseteq L_p(S) = L_{p+1}(S) = \cdots = \mathcal{L}(S)$. 
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A number $l(S)$ is called the length of the set $S$ provided

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$$l(S) = \min\{h \in \mathbb{Z}_+ : \mathcal{L}_h(S) = \mathcal{L}(S)\}.$$
Length is important in case we need to check whether all products in the given matrices share some property or satisfy given equations.

The general problem of evaluating
\[ l(M_n(\mathbb{F})) = \max\{l(S) : \mathcal{L}(S) = M_n(\mathbb{F})\} \]
as a function of \( n \) was posed in [Paz 1984] and has not been solved yet.
Length of the full matrix algebra

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Since \( \dim \mathcal{L}_i(S) < \dim \mathcal{L}_{i+1}(S) \) unless \( \mathcal{L}_i(S) = \mathcal{L}(S) \), then the trivial upper bound for the length is \( l(S) \leq n^2 - 1 \).
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**Conjecture ([Paz 1984])**

\[ l(S) \leq 2n - 2. \]

It is true for \( n \leq 4 \).
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Commutative and quasi-commutative sets

For commutative matrix sets we have


If $S$ is a commutative set, then $l(S) \leq n - 1$.

Quasi-commutativity (as a generalization of commutativity):

**Definition**

If $A, B$ in $M_n(\mathbb{F})$ is such a pair that $AB$ and $BA$ is a linearly dependent set we say that $A$ and $B$ quasi-commute.

If the given factor $\omega \in \mathbb{F}$ in quasi-commutativity relation $AB = \omega BA$ is important for us, we say that $A, B$ commute up to a factor $\omega$ (or $\omega$-commute).
For commutative matrix sets we have

**Theorem ([Paz, 1984] for \( \mathbb{C} \), [Guterman, Markova, 2009] for all fields)**

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Quasi-commutativity (as a generalization of commutativity):

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From now on let $S_q = \{A_1, \ldots, A_s\}$ be a set of $n \times n$ matrices such that any pair of its elements quasi-commute.

**Theorem [Constantine, Darnall 2005]**

\[
l(S_q) \leq 2n - 2 \quad \text{if} \quad \mathcal{L}(S_q) = M_n(\mathbb{F});
\]
\[
l(S_q) \leq 2n - 3 \quad \text{if} \quad \mathcal{L}(S_q) \subseteq M_n(\mathbb{F}).
\]
Further on we consider the case of pairs $S_q = \{A, B\}$ with $AB = \omega BA$ and answer the question what are the possible values of $l(S_q)$ depending on the quasi-commutativity factor $\omega$?

First we present results from

Our aim

Further on we consider the case of pairs $S_q = \{A, B\}$ with $AB = \omega BA$ and answer the question what are the possible values of $l(S_q)$ depending on the quasi-commutativity factor $\omega$? First we present results from

Nilpotent product case

Theorem

Let $S_q = \{A, B\}$, where $A, B \in M_n(\mathbb{F})$ satisfy one of the following conditions:

I) $AB = BA$,

II) $(AB)^n = (BA)^n = 0$ and $AB = \alpha BA$ for some $\alpha \in \mathbb{F} \setminus \{0\}$,

III) $AB = 0$ or $BA = 0$.

Then $l(S_q) \leq n - 1$.

Moreover, for any $k = 1, \ldots, n - 1$ each of these three classes contains a pair of matrices with length $k$.

Example

Fix $a \in \mathbb{F}$. Let $J_n = J_n(0)$, $D_n = \text{diag}\{1, a, \ldots, a^{n-1}\}$. Then $J_n D_n = a D_n J_n$ and $l(\{J_n, D_n\}) = n - 1$, since $J_n^{n-1} = E_{1,n}$, while $(J_n^r D_n^t)_{1,n} = 0$ for $r \leq n - 2$, thus $J_n^{n-1} \notin \mathcal{L}_{n-2}(\{J_n, D_n\})$. 
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Non-nilpotent product case: preliminaries

Observation

If $AB = \omega BA$, $(AB)^n \neq 0$, then equating characteristic polynomials of $AB$ and $BA$, we get $\omega^{n-j} = 1$ whenever the coefficient $c_j$ of $\chi_{AB}(t)$ is non-zero. Therefore, $\omega$ is a root of unity of degree $\leq n$.

Therefore we consider $\varepsilon$-commuting pairs over fields containing required root of unity $\varepsilon$.

By a primitive root of unity $\varepsilon_k \in \mathbb{F}$ of order $k$ we mean a $k$-th root of unity such that $\varepsilon_k^m \neq 1$ whenever $1 \leq m < k$. 

A.E. Guterman, O.V. Markova, V. Mehrmann
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Theorem [Drazin 1951]

Let $\mathbb{F} = \overline{\mathbb{F}}$ and let $n \in \mathbb{N}$, $n \geq 2$. If matrices $A, B \in M_n(\mathbb{F})$ satisfy $AB = \varepsilon BA$ for some $\varepsilon \in \mathbb{F}$, $\varepsilon \neq 1$ and the matrix $AB$ is not nilpotent, then there exists an integer $0 \leq r \leq n - 2$ and an invertible matrix $P \in M_n(\mathbb{F})$ such that

$$P^{-1}AP = \begin{bmatrix} S & X \\ O & A_r \end{bmatrix}, \quad P^{-1}BP = \begin{bmatrix} T & Y \\ O & B_r \end{bmatrix},$$

(1)

where $\varepsilon$ is necessarily a primitive root of unity of order $k > 1$ dividing $n - r$, $S$ and $T$ are triangular $r \times r$ matrices, $ST$ and $TS$ are both nilpotent.
Drazin’s form for $\varepsilon$-commuting pair

Theorem [Drazin 1951]

\[ A_r = \begin{bmatrix} C & O & \ldots & O \\ O & \varepsilon C & \ldots & O \\ & \ddots & \ddots & \ddots \\ O & O & \ldots & \varepsilon^{k-1} C \end{bmatrix}, \quad B_r = \begin{bmatrix} O & O & \ldots & O & D_1 \\ D_2 & O & \ldots & O & 0 \\ & \ddots & \ddots & \ddots & \ddots \\ O & O & \ldots & D_k & 0 \end{bmatrix}, \]

where $C \in M_{(n-r)/k}(\mathbb{F})$ is a nonsingular matrix, such that $\sigma(C) \cap \varepsilon^j \sigma(C) = \emptyset$ for all $j = 1, \ldots, k-1$, and $D_1, \ldots, D_k \in M_{(n-r)/k}(\mathbb{F})$ are arbitrary nonsingular matrices satisfying the relations $D_i C = C D_i, \; i = 1, 2, \ldots, k$. 

A.E. Guterman, O.V. Markova, V. Mehrmann
For non-nilpotent product pairs our aim can be rephrased as:

**Question**

Replace “?” in the following table with “+” for realizable value, “−” for non-realizable for each possible order of the root of unity $\varepsilon$:

<table>
<thead>
<tr>
<th>$\frac{l(S)}{\text{ord}(\varepsilon)}$</th>
<th>0</th>
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<th>$2n - 2$</th>
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Even numbers

Any even number in \([2, 2n - 2]\) can be realized as the length of \(\varepsilon\)-commuting pair with its own appropriate root of unity.

Example

\[
C_k = \begin{bmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 \\
1 & 0 & 0 & \ldots & 0
\end{bmatrix}, \quad D_k = \begin{bmatrix}
1 & 0 & \ldots & 0 \\
0 & \varepsilon_k & \ldots & 0 \\
0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \varepsilon_{k-1}
\end{bmatrix} \in M_k(\mathbb{F}),
\]

\[
A_{k,n} = C_k \oplus O_{n-k}, \quad B_{k,n} = D_k \oplus O_{n-k} \in M_n(\mathbb{F}),
\]

\(S_k = \{A_{k,n}, B_{k,n}\}\). Then \(A_{k,n}B_{k,n} = \varepsilon_k B_{k,n}A_{k,n}\) and \(l(S_k) = 2k - 2\).
Any even number in $[2, 2n - 2]$ can be realized as the length of $\varepsilon$-commuting pair with its own appropriate root of unity.

**Example**

$$C_k = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 1 & 0 & 0 & \ldots & 0 \end{bmatrix}, \quad D_k = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & \varepsilon_k & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \varepsilon_k^{k-1} \end{bmatrix} \in M_k(\mathbb{F}),$$

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Are there realizable odd numbers?

1. The numbers 1 and $2n - 3$ are not realizable (for any root of unity).
2. The number $2n - 5$ is realizable for $n = 4$ with $\omega = -1$ and for $n = 6$ with $\omega = \varepsilon_3$ and is not realizable for $n = 5$ or $n > 6$. 
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Approaches

**Observation**

Since a set of vectors over $\mathbb{F}$ is linearly dependent if and only if it is linearly dependent over any extension of $\mathbb{F}$, then the length of a set over the algebraic closure of a field is equal to that of the original field. Therefore with no lack of generality in the proofs we may use all results valid in algebraically closed fields.

**Definition**

A matrix $A \in M_n(\mathbb{F})$ is called non-derogatory if $\dim \langle I, A, \ldots, A^{n-1} \rangle = n$, otherwise it is called derogatory.
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**Approaches**

1. For $r \neq 0$, $l(\{A, B\}) \leq r + l(\{A_r, B_r\})$.

2. Why non-derogatory matrix $A_r$ (non-derogatory matrix $C$) is special here: in this case the centralizer of $C$ is $F[C]$, hence each block of $B_r$ is a polynomial in $C$. Unfortunately, doesn’t work in derogatory case.
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3. Use of the block structure: the location of (non-)zero blocks of $B_r^i$ and $B_r^j$ is the same iff $i \equiv j \pmod{k}$. 
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Bounds for the length

Using these methods we obtained

**Theorem**

Let $k, n \in \mathbb{N}$, $r \in \mathbb{Z}_+$, $n \geq 4$, $r < n$, $1 < k < n$, $k | (n - r)$.
Consider $A, B \in M_n(\mathbb{F})$ satisfying $AB = \varepsilon_k BA$ and $AB$ has an eigenvalue 0 of multiplicity $r$. Then

1. $2k - 2 \leq l(\{A, B\})$;
2. if $A_r$ and/or $B_r$ is non-derogatory, then $l(\{A, B\}) \leq n + k - 2$;
3. for each $k | n$ the value $n + k - 2$ is realizable.

For derogatory case the bound was $l(\{A, B\}) \leq 2(n - k) - r - 2$. 
Realizable values

**Theorem**

For all natural numbers $k, r, r \geq k \geq 2$ and any $n \geq r + k$ there exist matrices $A_n, B_n \in M_n(\mathbb{F})$ such that $A_nB_n = \varepsilon_k B_n A_n$, $(A_nB_n)^n \neq 0$, and $l(\{A_n, B_n\}) = r + k - 1$.

Consequently, for each $l \in \{2k - 2, \ldots, n - 1\}$ there exist matrices $A_l, B_l \in M_n(\mathbb{F})$ such that $(A_lB_l)^n \neq 0$, $A_lB_l = \varepsilon_k B_l A_l$ and $l(\{A_l, B_l\}) = l$.

**Example (Construction for previous Theorem)**

$$A_n = \text{O}_{n-r-k} \oplus J_r(0) \oplus D_k,$$
$$B_n = \text{I}_{n-r-k} \oplus D_r \oplus C_k,$$
where

$$D_t = \text{diag}\{1, \varepsilon_k, \varepsilon_k^2, \ldots, \varepsilon_k^{t-1}\} \in M_t(\mathbb{F})$$
and

$$C_k = E_{1,k} + E_{2,1} + \ldots E_{k,k-1} \in M_k(\mathbb{F}).$$
Main definitions
Lengths of matrix sets
Lengths of quasi-commutative sets

Realizable values

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Example (Construction for previous Theorem)

\[
A_n = O_{n-r-k} \oplus J_r(0) \oplus D_k, \\
B_n = I_{n-r-k} \oplus D_r \oplus C_k, \text{ where} \\
D_t = \text{diag}\{1, \varepsilon_k, \varepsilon_k^2, \ldots, \varepsilon_k^{t-1}\} \in M_t(\mathbb{F}) \text{ and} \\
C_k = E_{1,k} + E_{2,1} + \ldots + E_{k,k-1} \in M_k(\mathbb{F}).
\]
New tighter bounds for the length

Using the fact that the spanning set for $\mathcal{L}({A_r, B_r})$ can be chosen by taking a basis of the commutative algebra $\mathcal{L}({A_r, B_r^k})$ and multiplying it by all $B_i^i, i = 0, \ldots, k - 1$ and a more detailed application of results in commutative case we recently obtained a better bound for derogatory case and also improved the non-derogatory result:

**Theorem (2017)**

Let $k, n \in \mathbb{N}, r \in \mathbb{Z}_+, n \geq 4, r < n, 1 < k < n, k|(n − r)$. Consider $A, B \in M_n(\mathbb{F})$ satisfying $AB = \varepsilon_k BA$ and $AB$ has an eigenvalue 0 of multiplicity $r$. If $A_r$ and $B_r$ (defined in (1)) are both derogatory, then $l({A, B}) \leq n − 2$. 

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Consider $A, B \in M_n(\mathbb{F})$ satisfying $AB = \varepsilon_k BA$ and $AB$ has an eigenvalue 0 of multiplicity $r$.
If $A_r$ and/or $B_r$ is non-derogatory, then
1. $l(\{A, B\}) \leq n - r + k - 2$ for $0 \leq r \leq k - 1$ and
   $l(\{A, B\}) \leq n - 1$ for $r \geq k$;
2. if $l(\{A, B\}) < n - r + k - 2$, then $l(\{A, B\}) \leq n - 2$. 

A.E. Guterman, O.V. Markova, V. Mehrmann
Realizability table in general form \((n = 12)\)

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THANK YOU!
Main definitions
Lengths of matrix sets
Lengths of quasi-commutative sets

References


