Length realizability problem for pairs of quasi-commuting matrices

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Main definitions

Let $\mathbb{F}$ be an arbitrary field and $n > 1$ be a fixed integer. Let $\mathcal{S} = \{A_1, \dots, A_s\}$ be a set of $n \times n$ matrices over $\mathbb{F}$. Let us denote

- $\mathcal{S}^0 = \{I_n\}$;
- $\mathcal{S}^m = \{A_{i_1} \cdots A_{i_t} | 0 \leq t \leq m\}$, i.e. the set of products of length not greater than m in A_i;
- $\mathcal{L}_m(\mathcal{S}) = \langle \mathcal{S}^m \rangle$;
- $\mathcal{L}(\mathcal{S})$ — the linear span of all products in elements from $\mathcal{S}$. $\mathcal{L}_0(\mathcal{S}) \subseteq \mathcal{L}_1(\mathcal{S}) \subseteq \ldots \subseteq \mathcal{L}_p(\mathcal{S}) = \mathcal{L}_{p+1}(\mathcal{S}) = \cdots = \mathcal{L}(\mathcal{S})$.

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A number $l(\mathcal{S})$ is called the *length* of the set $\mathcal{S}$ provided

$$l(\mathcal{S}) = \min\{h \in \mathbb{Z}_+ : \mathcal{L}_h(\mathcal{S}) = \mathcal{L}(\mathcal{S})\}.$$

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Length of the full matrix algebra

Length is important in case we need to check whether all products in the given matrices share some property or satisfy given equations.

The general problem of evaluating $l(M_n(\mathbb{F})) = \max\{l(\mathcal{S}) : \mathcal{L}(\mathcal{S}) = M_n(\mathbb{F})\}$ as a function of n was posed in [Paz 1984] and has not been solved yet.

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Since $\dim \mathcal{L}_i(\mathcal{S}) < \dim \mathcal{L}_{i+1}(\mathcal{S})$ unless $\mathcal{L}_i(\mathcal{S}) = \mathcal{L}(\mathcal{S})$, then the trivial upper bound for the length is $l(\mathcal{S}) \leq n^2 - 1$.

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Conjecture ([Paz 1984])

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Commutative and quasi-commutative sets

For commutative matrix sets we have

Theorem ([Paz, 1984] for $\mathbb{C}$, [Guterman, Markova, 2009] for all fields)

If $\mathcal{S}$ is a commutative set, then $l(\mathcal{S}) \leq n-1$.

Quasi-commutativity (as a generalization of commutativity):

Definition

If A, B in $M_n(\mathbb{F})$ is such a pair that AB and BA is a linearly dependent set we say that A and B *quasi-commute*.

If the given factor $\omega \in \mathbb{F}$ in quasi-commutativity relation $AB = \omega BA$ is important for us, we say that A, B *commute up to a factor* ω (or *ω-commute*).

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Upper bounds for the length of quasi-commutative sets

From now on let $\mathcal{S}_q = \{A_1, \ldots, A_s\}$ be a set of $n \times n$ matrices such that any pair of its elements quasi-commute.

Theorem [Constantine, Darnall 2005]

$l(\mathcal{S}_q) \leq 2n - 2$ if $\mathcal{L}(\mathcal{S}_q) = M_n(\mathbb{F})$;
$l(\mathcal{S}_q) \leq 2n - 3$ if $\mathcal{L}(\mathcal{S}_q) \subset M_n(\mathbb{F})$.

Our aim

Further on we consider the case of pairs $\mathcal{S}_q = \{A, B\}$ with $AB = \omega BA$ and answer the question what are the possible values of $l(\mathcal{S}_q)$ depending on the quasi-commutativity factor ω?

First we present results from

- A.E. Guterman, O.V. Markova, *The realizability problem for values of the length function for quasi-commuting matrix pairs*, Zap. Nauchn. Sem. POMI. **439** (2015) 59–73.
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Nilpotent product case

Theorem

Let $\mathcal{S}_q = \{A, B\}$, where $A, B \in M_n(\mathbb{F})$ satisfy one of the following conditions:
I) $AB = BA$,
II) $(AB)^n = (BA)^n = 0$ *and* $AB = \alpha BA$ *for some* $\alpha \in \mathbb{F} \setminus \{0\}$,
III) $AB = 0$ *or* $BA = 0$.
Then $l(\mathcal{S}_q) \leq n - 1$.
Moreover, for any $k = 1, \ldots, n-1$ *each of these three classes contains a pair of matrices with length* k.

Example

Fix $a \in \mathbb{F}$. Let $J_n = J_n(0)$, $D_n = \mathrm{diag}\{1, a, \ldots, a^{n-1}\}$. Then $J_n D_n = a D_n J_n$ and $l(\{J_n, D_n\}) = n - 1$, since $J_n^{n-1} = E_{1,n}$, while $(J_n^r D_n^t)_{1,n} = 0$ for $r \leq n-2$, thus $J_n^{n-1} \notin \mathcal{L}_{n-2}(\{J_n, D_n\})$.

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Non-nilpotent product case: preliminaries

Observation

If $AB = \omega BA$, $(AB)^n \neq 0$, then equating characteristic polynomials of AB and BA, we get $\omega^{n-j} = 1$ whenever the coefficient c_j of $\chi_{AB}(t)$ is non-zero. Therefore, ω is a root of unity of degree $\leq n$.

Therefore we consider ε-commuting pairs over fields containing required root of unity ε.

By a primitive root of unity $\varepsilon_k \in \mathbb{F}$ of order k we mean a k-th root of unity such that $\varepsilon_k^m \neq 1$ whenever $1 \leq m < k$.

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Drazin's form for ε-commuting pair

Theorem [Drazin 1951]

Let $\mathbb{F} = \bar{\mathbb{F}}$ and let $n \in \mathbb{N}$, $n \geq 2$. If matrices $A, B \in M_n(\mathbb{F})$ satisfy $AB = \varepsilon BA$ for some $\varepsilon \in \mathbb{F}$, $\varepsilon \neq 1$ and the matrix AB is not nilpotent, then there exists an integer $0 \leq r \leq n-2$ and an invertible matrix $P \in M_n(\mathbb{F})$ such that

$$P^{-1}AP = \begin{bmatrix} S & X \\ O & A_r \end{bmatrix}, \quad P^{-1}BP = \begin{bmatrix} T & Y \\ O & B_r \end{bmatrix}, \qquad (1)$$

where ε is necessarily a *primitive root of unity of order* $k > 1$ dividing $n - r$, S and T are triangular $r \times r$ matrices, ST and TS are both nilpotent.

Drazin's form for ε-commuting pair

Theorem [Drazin 1951]

$$A_r = \begin{bmatrix} C & O & \dots & O \\ O & \varepsilon C & \dots & O \\ & & \ddots & \\ O & O & \dots & \varepsilon^{k-1} C \end{bmatrix}, \quad B_r = \begin{bmatrix} O & O & \dots & O & D_1 \\ D_2 & O & \dots & O & O \\ & \ddots & & & \\ O & O & \dots & D_k & O \end{bmatrix}, \tag{2}$$

where $C \in M_{(n-r)/k}(\mathbb{F})$ is a nonsingular matrix, such that $\sigma(C) \cap \varepsilon^j \sigma(C) = \varnothing$ for all $j = 1, \dots, k-1$, and $D_1, \dots, D_k \in M_{(n-r)/k}(\mathbb{F})$ are arbitrary nonsingular matrices satisfying the relations $D_i C = C D_i$, $i = 1, 2, \dots, k$.

Length realizability problem for quasi-commuting pairs

For non-nilpotent product pairs our aim can be rephrased as:

Question

Replace "?" in the following table with "+" for realizable value, "−" for non-realizable for each possible order of the root of unity ε:

$\frac{l(\mathcal{S})}{\mathrm{ord}(\varepsilon)}$	0	1	…	$2n-2$
1	**?**	**?**	…	**?**
⋮	⋮	⋮	…	⋮
n	**?**	**?**	…	**?**

Even numbers

Any even number in $[2, 2n-2]$ can be realized as the length of ε-commuting pair with its own appropriate root of unity.

Example

$$C_k = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{bmatrix}, \quad D_k = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & \varepsilon_k & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \varepsilon_k^{k-1} \end{bmatrix} \in M_k(\mathbb{F}),$$

$A_{k,n} = C_k \oplus O_{n-k}$, $B_{k,n} = D_k \oplus O_{n-k} \in M_n(\mathbb{F})$, $\mathcal{S}_k = \{A_{k,n}, B_{k,n}\}$. Then $A_{k,n}B_{k,n} = \varepsilon_k B_{k,n}A_{k,n}$ and $l(\mathcal{S}_k) = 2k-2$.

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Are there realizable odd numbers?

1. The numbers 1 and $2n-3$ are not realizable (for any root of unity).
2. The number $2n-5$ is realizable for $n=4$ with $\omega=-1$ and for $n=6$ with $\omega=\varepsilon_3$ and is not realizable for $n=5$ or $n>6$.

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Approaches

Observation

Since a set of vectors over $\mathbb{F}$ is linearly dependent if and only if it is linearly dependent over any extension of $\mathbb{F}$, then the length of a set over the algebraic closure of a field is equal to that of the original field. Therefore with no lack of generality in the proofs we may use all results valid in algebraically closed fields.

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1. For $r \neq 0$, $l(\{A, B\}) \leq r + l(\{A_r, B_r\})$.
2. Why non-derogatory matrix A_r (non-derogatory matrix C) is special here: in this case the centralizer of C is $\mathbb{F}[C]$, hence each block of B_r is a polynomial in C. Unfortunately, doesn't work in derogatory case.

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Bounds for the length

Using these methods we obtained

Theorem

Let $k, n \in \mathbb{N}$, $r \in \mathbb{Z}_+$, $n \geq 4$, $r < n$, $1 < k < n$, $k|(n-r)$. *Consider* $A, B \in M_n(\mathbb{F})$ *satisfying* $AB = \varepsilon_k BA$ *and* AB *has an eigenvalue* 0 *of multiplicity* r. *Then*

1. $2k - 2 \leq l(\{A, B\})$;

2. if A_r *and/or* B_r *is* *non-derogatory*, *then* $l(\{A, B\}) \leq n + k - 2$;

3. for each $k|n$ *the value* $n + k - 2$ *is realizable.*

For derogatory case the bound was $l(\{A, B\}) \leq 2(n-k) - r - 2$.

Realizable values

Theorem

For all natural numbers k, r, $r \geq k \geq 2$ and any $n \geq r + k$ there exist matrices $A_n, B_n \in M_n(\mathbb{F})$ such that $A_n B_n = \varepsilon_k B_n A_n$, $(A_n B_n)^n \neq 0$, and $l(\{A_n, B_n\}) = r + k - 1$.
Consequently, for each $l \in \{2k - 2, \ldots, n - 1\}$ there exist matrices $A_l, B_l \in M_n(\mathbb{F})$ such that $(A_l B_l)^n \neq 0$, $A_l B_l = \varepsilon_k B_l A_l$ and $l(\{A_l, B_l\}) = l$.

Example (Construction for previous Theorem)

$A_n = O_{n-r-k} \oplus J_r(0) \oplus D_k$,
$B_n = I_{n-r-k} \oplus D_r \oplus C_k$, where
$D_t = \mathrm{diag}\{1, \varepsilon_k, \varepsilon_k^2, \ldots, \varepsilon_k^{t-1}\} \in M_t(\mathbb{F})$ and
$C_k = E_{1,k} + E_{2,1} + \ldots E_{k,k-1} \in M_k(\mathbb{F})$.

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$C_k = E_{1,k} + E_{2,1} + \ldots E_{k,k-1} \in M_k(\mathbb{F})$.

New tighter bounds for the length

Using the fact that the spanning set for $\mathcal{L}(\{A_r, B_r\})$ can be chosen by taking a basis of the commutative algebra $\mathcal{L}(\{A_r, B_r^k\})$ and multiplying it by all B^i, $i = 0, \ldots, k-1$ and a more detailed application of results in commutative case we recently obtained a better bound for derogatory case and also improved the non-derogatory result:

Theorem (2017)

Let $k, n \in \mathbb{N}$, $r \in \mathbb{Z}_+$, $n \geq 4$, $r < n$, $1 < k < n$, $k|(n-r)$. Consider $A, B \in M_n(\mathbb{F})$ satisfying $AB = \varepsilon_k BA$ and AB has an eigenvalue 0 of multiplicity r. If A_r and B_r (defined in (1)) are both derogatory, then
$l(\{A, B\}) \leq n - 2$.

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Gap in realizable values for a non-derogatory pair

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If A_r *and/or* B_r *is non-derogatory, then*
1. $l(\{A, B\}) \leq n - r + k - 2$ *for* $0 \leq r \leq k - 1$ *and* $l(\{A, B\}) \leq n - 1$ *for* $r \geq k$;
2. if $l(\{A, B\}) < n - r + k - 2$, *then* $l(\{A, B\}) \leq n - 2$.

Realizability table in general form ($n = 12$)

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
1	+	+	+	+	+	+	+	+	+	+	+	+	−	−	−	−	−	−	−	−	−	−	−
2	−	−	+	+	+	+	+	+	+	+	+	+	+	−	−	−	−	−	−	−	−	−	−
3	−	−	−	−	+	+	+	+	+	+	+	+	−	+	−	−	−	−	−	−	−	−	−
4	−	−	−	−	−	−	+	+	+	+	+	+	−	−	+	−	−	−	−	−	−	−	−
5	−	−	−	−	−	−	−	−	+	+	+	+	−	+	−	−	−	−	−	−	−	−	−
6	−	−	−	−	−	−	−	−	−	−	+	+	−	−	−	−	+	−	−	−	−	−	−
7	−	−	−	−	−	−	−	−	−	−	−	−	+	−	−	−	−	−	−	−	−	−	−
8	−	−	−	−	−	−	−	−	−	−	−	−	−	−	+	−	−	−	−	−	−	−	−
9	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	+	−	−	−	−	−	−
10	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	+	−	−	−	−
11	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	+	−	−
12	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	+

THANK YOU!

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