Matrix Problems in Quantum Information Science

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- Let $\mathrm{diag}\,(x_1, \ldots, x_n) = \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix}$.

A classical result

Theorem [Weyl, Lidskii, etc.]

Let $A, B \in M_n$ be Hermitian matrices with eigenvalues $a_1 \geq \cdots \geq a_n$ and $b_1 \geq \cdots \geq b_n$.

$$(a_1 - b_1, \ldots, a_n - b_n) \prec \lambda(A - B) \prec (a_1 - b_n, \ldots, a_n - b_1).$$

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The above result implies that

$$\max_{X \in \mathcal{U}(B)} \|A - X\| = \|\mathrm{diag}\,(a_1 - b_n, \ldots, a_n - b_1)\|,$$

and

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  Step 1. If $\Delta_1 \geq \cdots \geq \Delta_n$, then set $(d_1, \dots, d_n) = \lambda(A) - (\Delta_1, \dots, \Delta_n)$ and stop. Else, go to Step 2.

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  Step 2. Let $2 \le j < k \le \ell \le n$ be such that

  $$\Delta_{j-1} \ne \Delta_j = \cdots = \Delta_{k-1} < \Delta_k = \cdots = \Delta_\ell \ne \Delta_{\ell+1}.$$

  Replace each $\Delta_j, \ldots, \Delta_\ell$ by $(\Delta_j + \cdots + \Delta_\ell)/(\ell - j + 1)$, and go to Step 1.

Examples

The following two examples illustrate the algorithm in the theorem.

Example 1 Let $A = \frac{1}{10}\text{diag}\,(4, 3, 3, 0)$ and $B = \frac{1}{10}\text{diag}\,(3, 3, 3, 1)$.

Apply Step 0:

Set $(\Delta_1, \ldots, \Delta_4) = \frac{1}{10}\text{diag}\,(4, 3, 3, 0) - \frac{1}{10}\text{diag}\,(3, 3, 3, 1) = \frac{1}{10}\text{diag}\,(1, 0, 0, -1)$.

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Of course, it depends on the type of quantum operation applied.

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Mixed Unitary and Unital Channels

One may consider using mixed unitary channels of the form

$$\Phi(X) = \sum_{j=1}^{k} p_j U_j X U_j^*,$$

where $p_1, \ldots, p_k > 0$ summing up to 1, and $U_1, \ldots, U_k \in M_n$ are unitary.

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Theorem [Li and Poon, 2011]

Let $\rho, \sigma \in M_n$ be density matrices. The following are equivalent.

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Step 2. Let $2 \leq j < k \leq \ell \leq n$ be such that

$$\Delta_{j-1} \neq \Delta_j = \cdots = \Delta_{k-1} < \Delta_k = \cdots = \Delta_\ell \neq \Delta_{\ell+1}.$$

Replace each $\Delta_j, \ldots, \Delta_\ell$ by $(\Delta_j + \cdots + \Delta_\ell)/(\ell - j + 1)$, and go to Step 1.

Examples

The two previous examples illustrating the algorithm in the theorem.

Example 1 Let $\sigma = \frac{1}{10}\text{diag}\,(4,3,3,0)$ and $\rho = \frac{1}{10}\text{diag}\,(3,3,3,1)$.

Apply Step 0:

Set $(\Delta_1, \ldots, \Delta_4) = \frac{1}{10}\text{diag}\,(4,3,3,0) - \frac{1}{10}\text{diag}\,(3,3,3,1) = \frac{1}{10}\text{diag}\,(1,0,0,-1)$.

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Assume one can use any quantum channel

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If $k = r$, then exit. Else, replace r, a, b by $r - k, (a_{k+1}, \dots, a_r), (b_{k+1}, \dots, b_r)$ and go to Step 1.

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- We also considered general quantum channels to maximize the fidelity function.
- Results are also obtained results for other functions on two density matrices such as the relative entropy:

$$S(A||B) = \mathrm{tr}\, A(\log_2 A - \log_2 B).$$

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Thank you for your attention!

Hope that you are interested in the problems

and will solve some or all of them!

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