Linear spaces of symmetric nilpotent matrices

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Questions

Can a nonzero real symmetric matrix be nilpotent?

Can a nonzero complex symmetric matrix be nilpotent?

$$N = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}.$$

What is the possible rank of a symmetric nilpotent matrix?

What is the maximal possible dimension of a linear space consisting of symmetric nilpotents?

How does a linear space consisting of symmetric nilpotents look like?

Can a linear space consisting of symmetric nilpotents be trianguarizable?

Case $n = 3$

Can a 3×3 symmetric nilpotent matrix have rank 2?

$$N = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & i \\ 1 & i & 0 \end{pmatrix}$$

Every 3×3 symmetric nilpotent matrix of rank 2 is orthogonally simmilar to N, because of

Lemma

If two symmetric matrices A and B are similar, then they are orthogonally similar.

A linear space $\mathscr{L}$, spanned by N and N^2, consists of nilpotents.

$\mathscr{L}$ is a maximal linear space of symmetric nilpotents, and also an algebra.

Notation

Denote:
J_k – $k \times k$ Jordan block,
I_k – $k \times k$ identity matrix,
Q_k – $k \times k$ matrix with 1's on the anti-diagonal and 0's elsewhere,
E_{ij} – the matrix with 1 at the ijth position and 0's elsewhere.

Write $n = 2k$ for n even and $n = 2k + 1$ for n odd.

Let

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} Q_k & -Q_k i \\ I_k & I_k i \end{pmatrix} \text{ for } n \text{ even and}$$

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} Q_k & 0 & -Q_k i \\ 0 & \sqrt{2} & 0 \\ I_k & 0 & I_k i \end{pmatrix} \text{ for } n \text{ odd.}$$

Persymmetric matrices

- $SS^* = I_n$, S is unitary and invertible,
- $SS^T = Q_n$.

We call a matrix persymmetric, if it is symmetric with respect to the anti-diagonal.

A matrix A is persymmetric if and only if $A^T = QAQ$.

Lemma

The mapping $A \mapsto S^*AS$ defines a bijective linear correspondence from the linear space of all persymmetric matrices onto the linear space of all symmetric matrices.

Proof: For A persymmetric

$$(S^*AS)^T = S^T A^T (S^*)^T = S^T QAQ(S^*)^T = S^*AS,$$

since $SS^T = Q$ implies $S^* = S^T Q$ and $S = Q(S^*)^T$.

Persymmetric matrices

There is a natural way of recognizing nilpotent matrices among the persymmetric ones – the strictly upper triangular persymmetric matrices are nilpotent.

Corollary

In $n \times n$ complex matrices there exists a symmetric nilpotent of rank $n-1$.

$$S^* J_3 S = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & i \\ 0 & i & 0 \end{pmatrix}.$$

Gerstenhaber's theorem

Theorem (Gerstenhaber, 1958)

If $\mathscr{L}$ is a linear space of $n \times n$ nilpotent matrices, then $\dim \mathscr{L} \leq \frac{n(n-1)}{2}$. If $\dim \mathscr{L} = \frac{n(n-1)}{2}$, then $\mathscr{L}$ is simultaneously similar to the linear space of strictly upper triangular matrices.

Perorthogonal matrices

A matrix V perorthogonal, if $VQV^TQ = I$ or $V^{-1} = QV^TQ$.

Matrices A and B perorthogonally similar, if there is a perorthogonal matrix $V \in M_n(\mathbb{C})$ such that $B = VAQV^TQ$.

If two matrices A and B are perorthogonally similar and A is persymmetric, then B is persymmetric.

Lemma

If two persymmetric matrices A and B are similar, then they are perorthogonally similar.

Lemma

Let matrices A and B be persymmetric. Then they are perorthogonally similar if an only if their respective symmetric versions S^*AS and S^*BS are orthogonally similar.

Radjavi's theorem

Theorem (Radjavi, 1986)

A set $\mathscr{N}$ of nilpotent matrices is triangularizable if it has the property that whenever A and B are in $\mathscr{N}$, there is a noncommutative polynomial p such that $AB + p(A,B)A$ is in $\mathscr{N}$.

A set of symmetric matrices cannot be triangularizable (i.e. simultaneously similar to a set upper triangular matrices) without loosing their symmetricity.

A set of persymmetric matrices is pertriangularizable, if it is simultaneously perorthogonally similar to a set of upper triangular matrices.

Radjavi's theorem

Theorem

A set $\mathscr{N}$ of nilpotent persymmetric matrices is pertriangularizable if it has the property that whenever A and B are in $\mathscr{N}$, there is a noncommutative polynomial p such that $AB + p(A,B)A$ is in $\mathscr{N}$.

Corollary

A set $\mathscr{N}$ of nilpotent persymmetric matrices is pertriangularizable if whenever A and B are in $\mathscr{N}$, there is a scalar c such that $AB - cBA$ is in $\mathscr{N}$.

Corollary

A set $\mathscr{N}$ of nilpotent persymmetric matrices is pertriangularizable if it is closed under Lie products (i.e. $A, B \in \mathscr{N}$ implies $AB - BA \in \mathscr{N}$).

Standard basis

Let

$$T_r = E_{r,n+1-r}, \quad \text{for} \quad r = 1, 2, ...k$$

and

$$T_{rs} = E_{rs} + QE_{sr}Q = E_{rs} + E_{n+1-s,n+1-r}$$

for $r = 1, 2, ..., k$ and $s = r + 1, ..., n - r$.

Let

$$N_r = S^* T_r S, \quad N_{rs} = S^* T_{rs} S.$$

The set of matrices

$$\{T_r \,|\, r = 1, 2, ..., k\} \cup \{T_{rs} \,|\, r = 1, 2, ..., k, s = r + 1, ..., n - r\}$$

forms a basis of a vector space of strictly upper triangular pesymmeric matrices, of dimension $\left\lfloor \frac{n^2}{4} \right\rfloor$, which is also an algebra.

Standard basis

The set of matrices

$$\{N_r \mid r = 1,2,...,k\} \cup \{N_{rs} \mid r = 1,2,...,k, s = r+1,...,n-r\}$$

forms a basis of an algebra of nilpotent symmetric matrices of dimension $\left\lfloor \frac{n^2}{4} \right\rfloor$, denoted by $\mathscr{N}^{(0)}$.

Theorem

If $\mathscr{L}$ is a linear space of $n \times n$ symmetric nilpotent matrices, then $\dim \mathscr{L} \leqslant \left\lfloor \frac{n^2}{4} \right\rfloor$. If $\dim \mathscr{L} = \left\lfloor \frac{n^2}{4} \right\rfloor$, then $\mathscr{L}$ is simultaneously orthogonally similar to $\mathscr{N}^{(0)}$.

Increasing order by 4

Can we have a linear space of symmetric nilpotent matrices that is maximal in the usual sense (i.e. every strictly greater space of symmetric matrices contains a non-nilpotent), but has dimension smaller than $\left\lfloor \frac{n^2}{4} \right\rfloor$?

Let the dimension of the underlying space be $n+4$ for $n = 1, 2, \dots$ and divided into three consecutive blocks of respective dimensions 2, n, and 2. We fix a linear space $\mathscr{L}$ of persymmetric nilpotent matrices acting on $\mathbb{C}^n$ and define a space $\widehat{\mathscr{L}}$ of matrices, acting on $\mathbb{C}^{n+4}$, by

$$\widehat{\mathscr{L}} = \left\{ A; A = \begin{pmatrix} xE_{12} & xE_{21} & 0 \\ yE_{n1} & T & xE_{n1} \\ -2yE_{12} & yE_{21} & xE_{12} \end{pmatrix}, x, y \in \mathbb{C}, \in T \in \mathscr{L} \right\}.$$

5×5 matrices

In the case of 5×5 matrices we have

$$\widehat{\mathscr{L}} = \left\{ A; A = \begin{pmatrix} 0 & x & 0 & 0 & 0 \\ 0 & 0 & x & 0 & 0 \\ y & 0 & 0 & x & 0 \\ 0 & -2y & 0 & 0 & x \\ 0 & 0 & y & 0 & 0 \end{pmatrix}, x, y \in \mathbb{C} \right\}.$$

Proposition

The space $\widehat{\mathscr{L}}$ is a linear space of persymmetric nilpotents.

Maximality of $\widehat{\mathscr{L}}$

Let

$$X = \begin{pmatrix} E_{12} & E_{21} & 0 \\ 0 & 0 & E_{n1} \\ 0 & 0 & E_{12} \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ E_{n1} & 0 & 0 \\ -2E_{12} & E_{21} & 0 \end{pmatrix},$$

and

$$Z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Write any other persymmetric matrix with respect to this block partition as

$$\begin{pmatrix} D & U & W \\ V & T & Q_n U^T Q_2 \\ R & Q_2 V^T Q_n & Q_2 D^T Q_2 \end{pmatrix}. \tag{1}$$

Maximality of $\widehat{\mathscr{L}}$

Proposition

Suppose $\mathscr{T}$ is a linear set of persymmetric nilpotents containing X, Y, and Z, then any member of $\mathscr{T}$ is a linear combination of X, Y, and a matrix of the form (1) in which all block entries except T are zero.

Theorem

If the space $\mathscr{L}$ is a maximal linear space of symmetric nilpotent matrices on $\mathbb{C}^n$ in the sense that there is no strictly greater linear space of symmetric nilpotents and if it contains a nilpotent of rank $n-1$, then $\widehat{\mathscr{L}}$ is a maximal linear space of symmetric nilpotent matrices on $\mathbb{C}^{n+4}$ and it contains a nilpotent of rank $n+3$.

Small n

Let $\mathscr{L} \subseteq M_n(\mathbb{C})$ be a maximal linear space of symmetric nilpotent matrices.

If $n = 2$, then $\dim \mathscr{L} = 1$.

If $n = 3$, then $\dim \mathscr{L} = 2$.

If $n = 5$, then there exists $\mathscr{L}$ with $\dim \mathscr{L} = 2$.

If $n = 6$, then there exists $\mathscr{L}$ with $\dim \mathscr{L} = 3$.

Proposition

Let $\mathscr{L} \subseteq M_4(\mathbb{C})$ be a maximal linear space of symmetric nilpotent matrices. Then it is of dimension 4.

Case $n = 7$

Example

The set $\mathscr{L}_7$ of matrices of the form

$$A = \begin{pmatrix} 0 & x & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x & 0 & 0 & 0 & 0 \\ 2z & 0 & 0 & x & 0 & 0 & 0 \\ 0 & -z & 0 & 0 & x & 0 & 0 \\ y & 0 & -2z & 0 & 0 & x & 0 \\ 0 & -2y & 0 & -z & 0 & 0 & x \\ 0 & 0 & y & 0 & 2z & 0 & 0 \end{pmatrix}$$

is a maximal linear space of persymmetric nilpotent matrices of dimension 3.

Large n

Example

There exists $\mathscr{L}_8 \subseteq M_8(\mathbb{C})$ which is a maximal linear space of persymmetric nilpotent matrices of dimension 4.

Theorem

For every $n > 4$ there exists a maximal linear space $\mathscr{L}$ of symmetric nilpotent matrices on $\mathbb{C}^n$ such that $\dim \mathscr{L} = \left\lfloor \frac{n}{2} \right\rfloor$. The space contains a nilpotent of maximal rank.

Thank you!