On projections arising from isometries with finite spectrum on Banach spaces

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Connections to Slovenia

- L.L. Stachó, B. Zalar, *Bicircular projections on some matrix and operator spaces*, Linear Algebra Appl. 384 (2004), 21-42.
- L.L. Stachó, B. Zalar, *Bicircular projections and characterization of Hilbert spaces*, Proc. Amer. Math. Soc. 132 (2004), 3019-3025.

Spring Semester 2004 in Maribor, Slovenia:

- M. Fošner, D. Ilišević, *On a class of projections on $*$-rings*, Comm. Algebra 33 (2005), 3293–3310.

Connections to Slovenia

- L.L. Stachó, B. Zalar, *Bicircular projections on some matrix and operator spaces*, Linear Algebra Appl. 384 (2004), 21-42.
- L.L. Stachó, B. Zalar, *Bicircular projections and characterization of Hilbert spaces*, Proc. Amer. Math. Soc. 132 (2004), 3019-3025.

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- M. Fošner, D. Ilišević, *On a class of projections on* $*$*-rings*, Comm. Algebra 33 (2005), 3293–3310.

Connections to Slovenia

4th Linear Algebra Workshop, Bled, Slovenia, 2005

Pancakes at Plemelj's house

Connections to Slovenia

- M. Fošner, D. Ilišević and C. K. Li, *G-invariant norms and bicircular projections*, Linear Algebra Appl. 420 (2007), 596–608.

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Projections on Banach spaces

Definition

A **projection** $P\colon \mathcal{X} \to \mathcal{X}$ on a Banach space $\mathcal{X}$ is a bounded linear operator such that $P^2 = P$.

- $I - P$ is also a projection,
- $\|P\| \geq 1$,
- $P\mathcal{X}$ is a closed subspace of $\mathcal{X}$,
- $\mathcal{X} = P\mathcal{X} + (I - P)\mathcal{X}$.

Trivial projections: 0 and I.

Orthogonal projections on Hilbert spaces

Definition

An **orthogonal projection** $P\colon \mathcal{H} \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$ is a bounded self-adjoint projection, that is, $P^2 = P = P^*$.

- $I - P$ is also an orthogonal projection,
- $\|P\| = 1$,
- $P\mathcal{H}$ is a closed subspace of $\mathcal{H}$,
- $\mathcal{H} = P\mathcal{H} + (I-P)\mathcal{H}$ with $P\mathcal{H} \perp (I-P)\mathcal{H}$, which is equivalent to $\|x + \lambda y\| = \|x - \lambda y\|$ for all scalars λ and all vectors $x \in P\mathcal{H}$ and $y \in (I-P)\mathcal{H}$.

Non-orthogonal projections on Hilbert spaces

Example

Let $\mathcal{H}$ be a two-dimensional Hilbert space, and

$$P(x,y) = \left(x - \frac{y}{2}, 0\right).$$

Then P is a non-orthogonal projection on $\mathcal{H}$, and

$$(I-P)(x,y) = \left(\frac{y}{2}, y\right).$$

Generalized orthogonal projections

A bounded linear operator P on a Banach space is a projection if and only if $T = 2P - I$ is a reflection, that is, $T^2 = I$.

If P is an orthogonal projection on a Hilbert space then $U = 2P - I$ is a surjective linear isometry with spectrum $\sigma(U) = \{-1, 1\}$.

Then P and $I - P$ are the eigenprojections of U associated to the eigenvalues 1 and -1, respectively.

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Generalized orthogonal projections

Definition

A non-trivial projection P on a Banach space $\mathcal{X}$ is called a **generalized orthogonal projection** if P and $I - P$ are the eigenprojections of a (surjective) isometry T on $\mathcal{X}$ with $T^2 = I$ associated to its eigenvalues 1 and -1, respectively.

Notice that $T = P - (I - P)$ and $P = \frac{I+T}{2}$.

Generalized orthogonal projections

Two elements x, y in a Banach space $\mathcal{X}$ are said to be **Roberts orthogonal** if $\|x + \lambda y\| = \|x - \lambda y\|$ for all scalars λ.

Elements in $P\mathcal{X}$ are Roberts orthogonal to elements in $(I - P)\mathcal{X}$ if and only if P is a generalized orthogonal projection.

Generalized bicircular projections

Definition

A projection P on a Banach space $\mathcal{X}$ is called a **generalized bicircular projection** if there is a (surjective) isometry $T\colon \mathcal{X} \to \mathcal{X}$ with spectrum $\sigma(T) = \{e^{2\pi r\mathbf{i}}, e^{2\pi s\mathbf{i}}\}$ for some distinct real numbers r, s such that P and $I - P$ are eigenprojections of T associated to $e^{2\pi r\mathbf{i}}$ and $e^{2\pi s\mathbf{i}}$, respectively.

Replacing T with $e^{-2\pi r\mathbf{i}}T$, we can assume that $e^{2\pi r\mathbf{i}} = 1$.

Notice that

$$T = P + e^{2\pi s\mathbf{i}}(I - P) \text{ and } P = \frac{T - e^{2\pi s\mathbf{i}}I}{1 - e^{2\pi s\mathbf{i}}}.$$

When $e^{2\pi s\mathbf{i}} = -1$ then $P = \frac{I+T}{2}$ is a generalized orthogonal projection.

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Generalized bicircular projections

A projection P on a Hilbert space is self-adjoint if and only if $e^{2\pi t \mathbf{i} P}$ is unitary for all $t \in \mathbb{R}$.

Notice that

$$\begin{aligned} e^{2\pi t\,\mathbf{i}P} &= \sum_{n=0}^{\infty} \frac{(2\pi t\,\mathbf{i}P)^n}{n!} = I + P\left[\sum_{n=1}^{\infty} \frac{(2\pi t\,\mathbf{i})^n}{n!}\right] \\ &= I + P[e^{2\pi t\,\mathbf{i}} - 1] = e^{2\pi t\,\mathbf{i}}P + (I - P). \end{aligned}$$

Therefore, P is self-adjoint if and only if $P + e^{-2\pi t \mathbf{i}}(I - P)$ is a surjective isometry for all $t \in \mathbb{R}$.

Hermitian and bicircular projections

Definition

A projection P on a Banach space $\mathcal{X}$ is called a **hermitian projection** if $e^{2\pi t\mathbf{i}P}$ is a (surjective) isometry for all $t \in \mathbb{R}$, and it is called a **bicircular projection** if $P + e^{2\pi t\mathbf{i}}(I - P)$ is a (surjective) isometry for all $t \in \mathbb{R}$.

Theorem (J. Jamison, LAA, 2007)

A projection on $\mathcal{X}$ is a bicircular projection if and only if it is a hermitian projection.

Hermitian and bicircular projections

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Theorem (J. Jamison, LAA, 2007)

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Hermitian and generalized bicircular projections

Suppose that $T = P + e^{2\pi t\mathbf{i}}(I - P)$ is an isometry for some $t \in \mathbb{R}$.
Then $T^n = P + e^{2n\pi t\mathbf{i}}(I - P)$ is also an isometry for all $n \in \mathbb{N}$.
For $t \in \mathbb{R} \setminus \mathbb{Q}$ the set $\{e^{2n\pi t\mathbf{i}} : n \in \mathbb{N}\}$ is dense in the complex unit circle.
This implies that $P + e^{2\pi t\mathbf{i}}(I - P)$ is an isometry for all $t \in \mathbb{R}$.
Thus P is hermitian.
Therefore, the study of generalized bicircular projections emphasis on those associated to rational angles (P.-K. Lin, JMAA, 2008).

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Projections on Hilbert spaces

In the Hilbert space setting the following notions coincide:

- orthogonal projections,
- generalized orthogonal projections,
- generalized bicircular projections,
- hermitian (bicircular) projections.

Bicontractive projections

Definition

A projection P on a Banach space $\mathcal{X}$ is said to be **bicontractive** if $\|P\| = \|I - P\| = 1$.

Example

- Every orthogonal projection on a Hilbert space.
- Every generalized orthogonal projection P since $P = \frac{I+T}{2}$ for some isometry T.
- Every generalized bicircular projection (P.-K. Lin, JMAA, 2008).
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- In particular, every hermitian (bicircular) projection.

Hermitian projections and bicontractive projections

Example (L.L. Stachó and B. Zalar, PAMS, 2004)

Let $M_2(\mathbb{C})$ be equipped with the spectral norm and let $P\colon M_2(\mathbb{C}) \to M_2(\mathbb{C})$ be defined by

$$P\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & \delta \end{bmatrix}.$$

Then P is a bicontractive projection. However, for $x = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ we have $\|x\| = 2$, but $\|(P + \mathbf{i}(I - P))(x)\| = \sqrt{2}$.

Hence $P + \mathbf{i}(I - P)$ is not an isometry and P is not a hermitian (bicircular) projection.

JB*-triples

A **JB*-triple** is a complex Banach space $\mathcal{A}$ together with a continuous triple product $\{\cdot\cdot\cdot\}\colon \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ such that

- **(i)** $\{xyz\}$ is linear in x and z and conjugate linear in y;
- **(ii)** $\{xyz\} = \{zyx\}$;
- **(iii)** for any $x \in \mathcal{A}$, the operator $\delta(x)\colon \mathcal{A} \to \mathcal{A}$ defined by $\delta(x)y = \{xxy\}$ is hermitian with nonnegative spectrum;
- **(iv)** $\delta(x)\{abc\} = \{\delta(x)a, b, c\} - \{a, \delta(x)b, c\} + \{a, b, \delta(x)c\}$;
- **(v)** for every $x \in \mathcal{A}$, $\|\{xxx\}\| = \|x\|^3$.

JB*-triples

Example

- complex Hilbert spaces: $\{xyz\} = \frac{1}{2}(\langle x, y\rangle z + \langle z, y\rangle x)$
- C*-algebras: $\{xyz\} = \frac{1}{2}(xy^*z + zy^*x)$

For every bicontractive linear projection $P\colon \mathcal{A} \to \mathcal{A}$ there exists a (surjective) linear isometry $T\colon \mathcal{A} \to \mathcal{A}$ such that

$$P = \frac{I + T}{2}$$

(Y. Friedman, B. Russo, Math. Z., 1987).

Generalized bicircular projections on JB*-triples

Theorem (D. I., LAA, 2010)

Let $\mathcal{A}$ be a JB-triple and let $P\colon \mathcal{A} \to \mathcal{A}$ be a rank one linear projection. Then P is bicontractive if and only if P is hermitian (bicircular).*

Theorem (D. I., LAA, 2010)

Let $\mathcal{A}$ be a JB-triple and let $P\colon \mathcal{A} \to \mathcal{A}$ be a linear projection. Then $P + e^{2\pi t\mathbf{i}}(I - P)$ is an isometry for some $t \in \mathbb{R}$ if and only if one of the following holds.*

(i) *P is hermitian (bicircular).*

(ii) *$e^{2\pi t\mathbf{i}} = -1$ and $P = \frac{1}{2}(I + T)$ for some linear isometry $T\colon \mathcal{A} \to \mathcal{A}$ satisfying $T^2 = I$.*

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(i) *P is hermitian (bicircular).*

(ii) *$e^{2\pi t\mathbf{i}} = -1$ and $P = \frac{1}{2}(I + T)$ for some linear isometry $T\colon \mathcal{A} \to \mathcal{A}$ satisfying $T^2 = I$.*

Applications to $C_0(\Omega)$

Corollary

Let $C_0(\Omega)$ be the algebra of all continuous complex-valued functions on a locally compact Hausdorff space Ω, vanishing at infinity, and let $P\colon C_0(\Omega) \to C_0(\Omega)$ be a linear projection. Then $P + e^{2\pi t\mathbf{i}}(I - P)$ is an isometry for some $t \in \mathbb{R}$ if and only if one of the following holds.

(i) *P is hermitian (bicircular).*

(ii) *$e^{2\pi t\mathbf{i}} = -1$ and there exist a homeomorphism $\varphi\colon \Omega \to \Omega$ satisfying $\varphi^2 = I$ and a continuous function $u\colon \Omega \to \mathbb{C}$ satisfying $|u(\omega)| = 1$ and $u(\varphi(\omega)) = \overline{u(\omega)}$ for every $\omega \in \Omega$, such that*

$$P(f)(\omega) = \frac{1}{2}\Big(f(\omega) + u(\omega)f\left(\varphi(\omega)\right)\Big), \quad \forall f \in C_0(\Omega),\, \omega \in \Omega.$$

Generalized bicircular projections on JB*-triples

Let $\mathcal{A}$ be a JB*-triple. We shall say that a subtriple $\mathcal{A}_1$ is **complementary** to $\mathcal{A}_2$ if $\ker(\mathcal{A}_1) = \mathcal{A}_2$ and $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2$. Here,

$$\ker(\mathcal{A}_1) := \{y \in \mathcal{J} : \{x, y, z\} = 0, \forall x, z \in \mathcal{A}_1\}$$

is an inner ideal of $\mathcal{A}$. Notice that $\mathcal{A}_1 \cap \ker(\mathcal{A}_1) = \{0\}$.

Theorem (D.I., C.-N. Liu, N.-C. Wong, Concr. Oper., 2017)

Let P be a generalized bicircular projection on a JB-triple $\mathcal{A}$. Then P is a generalized orthogonal projection, and $\mathcal{A} = P\mathcal{A} + (I - P)\mathcal{A}$ is a direct sum of JB*-subtriples. Furthermore, P is hermitian if and only if $P\mathcal{A}$ and $(I - P)\mathcal{A}$ are complementary to each other.*

Generalized n-circular projections

Definition

Let P_0 be a non-zero projection on a Banach space $\mathcal{X}$, and $n \geq 2$. We call P_0 a **generalized *n*-circular projection** if there exists a (surjective) isometry $T \colon \mathcal{X} \to \mathcal{X}$ with $\sigma(T) = \{1, \lambda_1, \ldots, \lambda_{n-1}\}$ consisting of n distinct modulus one eigenvalues such that P_0 is the eigenprojection of T associated to $\lambda_0 = 1$.
In this case, there are non-zero projections $P_1, \ldots, P_{n-1}$ on $\mathcal{X}$ such that

$$P_0 \oplus P_1 \oplus \cdots \oplus P_{n-1} = I \quad \text{and} \quad T = P_0 + \lambda_1 P_1 + \cdots + \lambda_{n-1} P_{n-1}.$$

We also say that P_0 is a generalized n-circular projection associated with $(\lambda_1, \ldots, \lambda_{n-1}, P_1, \ldots, P_{n-1})$.
We call P_0 a **proper** generalized n-circular projection $(n \geq 3)$ if it is not a generalized k-circular projection for any integer $1 < k < n$.

Generalized n-circular projections and (bi)contractivity

Lemma (D.I., Contemp. Math., 2017)

Every generalized n-circular projection on a complex Banach space is a contraction.

Remark

A proper generalized n-circular projection on a JB-triple is not bicontractive.*

Generalized n-circular projections and (bi)contractivity

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Remark

A proper generalized n-circular projection on a JB-triple is not bicontractive.*

Generalized n-circular projections on JB*-triples

Theorem (D. I., Contemp. Math., 2017)

Let $\mathcal{A}$ be a JB-triple, and $P_0\colon \mathcal{A} \to \mathcal{A}$ be a generalized n-circular projection, $n \geq 2$, associated with $(\lambda_1, \dots, \lambda_{n-1}, P_1, \dots, P_{n-1})$. Let $\lambda_0 = 1$. Then one of the following holds.*

- **(i)** *There exist $i, j, k \in \{0, 1, \dots, n-1\}$, $j \neq i$, $j \neq k$, such that $\lambda_i \overline{\lambda_j} \lambda_k \in \{\lambda_m : m = 0, 1, \dots, n-1\}$.*
- **(ii)** *All P_0, P_1, $\dots$, P_{n-1} are hermitian (bicircular).*

When $n = 2$: if P is not hermitian then $\lambda^2 \in \{1, \lambda\}$, or $\overline{\lambda} \in \{1, \lambda\}$; hence $\lambda = -1$.

Generalized n-circular projections on JB*-triples

Theorem (D. I., Contemp. Math., 2017)

Let $\mathcal{A}$ be a JB-triple, and $P_0 \colon \mathcal{A} \to \mathcal{A}$ be a generalized n-circular projection, $n \geq 2$, associated with $(\lambda_1, \ldots, \lambda_{n-1}, P_1, \ldots, P_{n-1})$. Let $\lambda_0 = 1$. Then one of the following holds.*

- **(i)** *There exist $i, j, k \in \{0, 1, \ldots, n-1\}$, $j \neq i$, $j \neq k$, such that $\lambda_i \overline{\lambda_j} \lambda_k \in \{\lambda_m : m = 0, 1, \ldots, n-1\}$.*
- **(ii)** *All P_0, P_1, $\ldots$, P_{n-1} are hermitian (bicircular).*

When $n = 2$: if P is not hermitian then $\lambda^2 \in \{1, \lambda\}$, or $\overline{\lambda} \in \{1, \lambda\}$; hence $\lambda = -1$.

Generalized tricircular projections on JB*-triples

Corollary (D. I., Contemp. Math., 2017)

Let $\mathcal{A}$ be a JB-triple, and $P\colon \mathcal{A} \to \mathcal{A}$ be a generalized tricircular projection associated with $(\lambda_1, \lambda_2, Q, R)$. Then one of the following holds.*

(i) $\lambda_1\lambda_2 = 1$, *or* $\lambda_1^2 = \lambda_2$, *or* $\lambda_1 = \lambda_2^2$.

(ii) *P, Q, R are hermitian (bicircular).*

Generalized n-circular projections on $C_0(\Omega)$

Let Ω be a locally compact Hausdorff space.
Let $\varphi\colon \Omega \to \Omega$ be a homeomorphism with period m, i.e., $\varphi^m = id_\Omega$ and $\varphi^k \neq id_\Omega$ for $k = 1, 2, \ldots, m-1$.
Let u be a continuous unimodular scalar function on Ω such that

$$u(\omega) \cdots u(\varphi^{m-1}(\omega)) = 1, \quad \forall \omega \in \Omega.$$

Then the surjective isometry $T\colon C_0(\Omega) \to C_0(\Omega)$ defined by

$$Tf(\omega) = u(\omega) f(\varphi(\omega))$$

satisfies $T^m = I$.
Therefore, the spectrum $\sigma(T) = \{\lambda_0, \lambda_1, \ldots, \lambda_{n-1}\}$ consists of n distinct mth roots of unity.
Replacing T with $\overline{\lambda_0} T$, we can assume that $\lambda_0 = 1$.

Generalized n-circular projections on $C_0(\Omega)$

This gives rise to a spectral decomposition

$$\begin{aligned} I &= P_0 + P_1 + \cdots + P_{n-1}, \\ T &= \lambda_0 P_0 + \lambda_1 P_1 + \cdots + \lambda_{n-1} P_{n-1}. \end{aligned}$$

Here, the spectral projections are defined by

$$\begin{aligned} P_i f(w) =& \frac{(I + \overline{\lambda_i} T + \cdots + \overline{\lambda_i}^{m-1} T^{m-1}) f(\omega)}{m} \\ =& \frac{1}{m}\Big(f(\omega) + \overline{\lambda_i} u(\omega) f\left(\varphi(\omega)\right) + \ldots \\ & + \overline{\lambda_i}^{m-1} u(\omega) \ldots u\left(\varphi^{m-2}(\omega)\right) f\left(\varphi^{m-1}(\omega)\right)\Big) \end{aligned}$$

for all $f \in C_0(\Omega)$, $\omega \in \Omega$, and $i = 0, 1, \ldots, n-1$.
An mth root λ of unity does not belong to $\sigma(T)$ if and only if

$$I + \overline{\lambda} T + \cdots + \overline{\lambda}^{m-1} T^{m-1} = 0.$$

Generalized n-circular projections on $C_0(\Omega)$

Theorem (D.I., C.-N. Liu, N.-C. Wong)

Let Ω be a locally compact space. Let T be a surjective isometry of $C_0(\Omega)$ with finite spectrum consisting of n points. Assume there is an eigenprojection of T being a proper n-circular projection, or Ω is connected. Then all eigenvalues of T are of finite orders.

Definition

We call the generalized n-circular projection P_0 **periodic** (resp. **primitive**) if it is an eigenprojection of a periodic surjective isometry T of period $m \geq n$ (resp. of period $m = n$).

Generalized 4-circular projections on $C_0(\Omega)$ – an example

Example (D.I., C.-N. Liu, N.-C. Wong)

$$A = \{(x,y,z) \in \mathbb{R}^3 : x,y,z \in [0,1]\},$$
$$B = \{(s,-s,0) \in \mathbb{R}^3 : s \in [-1,1]\}, \quad \Omega = A \cup B.$$

$$\varphi(x,y,z) = \begin{cases} (y,z,x), & \text{if } (x,y,z) \in A; \\ (-x,-y,-z), & \text{if } (x,y,z) \in B. \end{cases}$$

The isometry $Tf \overset{def}{=} f \circ \varphi$ of period 6 has 4 eigenvalues

$$\lambda_0 = 1, \ \lambda_1 = -1, \ \lambda_2 = \beta, \ \lambda_3 = \beta^2, \qquad \text{where } \beta = e^{\mathbf{i}\frac{2\pi}{3}}.$$

Hence

$$T = P_0 - P_1 + \beta P_2 + \beta^2 P_3.$$

Generalized n-circular projections on $C_0(\Omega)$ and bicontractivity

Notice that $T^3 = I - 2P_1$, which implies that the eigenprojection P_1 is bicontractive.

Theorem (D.I., C.-N. Liu, N.-C. Wong)

Let Ω be a connected locally compact Hausdorff space and let T be an isometry of $C_0(\Omega)$ that has odd period. Then none of the eigenprojections of T is bicontractive.

Generalized n-circular projections on $C_0(\Omega)$ – the structure theorem

Theorem (D.I., C.-N. Liu, N.-C. Wong)

Let Ω be a connected locally compact Hausdorff space.
Let $\varphi\colon \Omega \to \Omega$ be a homeomorphism and u be a unimodular continuous scalar function defined on Ω.
Let P_0 be a generalized n-circular projection on $C_0(\Omega)$ associated to the surjective isometry $Tf = u \cdot f \circ \varphi$ with the spectral decomposition

$$I = P_0 + P_1 + \cdots + P_{n-1},$$
$$T = P_0 + \lambda_1 P_1 + \cdots + \lambda_{n-1} P_{n-1}.$$

Assume all eigenvalues $\lambda_0 = 1, \lambda_1, \ldots, \lambda_{n-1}$ of T have a (minimum) finite common period $m \geq n$.
In particular, all of them are mth roots of unity, and $T^m = I$.
Then the following holds.

Generalized n-circular projections on $C_0(\Omega)$ – the structure theorem

Theorem (continuation)

- *The homeomorphism φ has (minimum) period m.*
- *The cardinality $k(\omega)$ of the orbit $\{\omega, \varphi(\omega), \varphi^2(\omega), \ldots\}$ of each point ω under φ is not greater than n.*
- *m is the least common multiple of $k(\omega)$ for all ω in Ω.*

Generalized n-circular projections on $C_0(\Omega)$ – the structure theorem

Theorem (continuation)

- *The spectrum $\sigma(T)$ of T can be written as a union of the complete set of $k(\omega)$th roots of the modulus one scalar $\alpha_\omega = u(\omega)u(\varphi(\omega))\cdots u(\varphi^{k(\omega)-1}(\omega))$. More precisely,*

$$\sigma(T) = \bigcup_{\omega\in\Omega} \{\lambda_\omega, \lambda_\omega\eta_\omega, \lambda_\omega\eta_\omega^2, \ldots, \lambda_\omega\eta_\omega^{k(\omega)-1}\},$$

  where λ_ω and η_ω are primitive $k(\omega)$th roots of α_ω and unity, respectively. We call the set in the union a complete cycle of $k(\omega)$th roots of unity shifted by λ_ω.
- *If $u(\omega) = 1$ on Ω then we can choose all $\lambda_\omega = 1$, and thus $\sigma(T)$ consists of all $k(\omega)$th roots of unity.*
- *If m is a prime integer, then $n = m$ and $\sigma(T)$ consists of the complete cycle of nth roots of unity.*

Generalized bicircular and tricircular projections on $C_0(\Omega)$

Corollary (D.I., C.-N. Liu, N.-C. Wong)

Let Ω be a connected locally compact Hausdorff space. Then every generalized bicircular or tricircular projection P_0 on $C_0(\Omega)$ is primitive. In other words, P_0 can only be an eigenprojection of a surjective isometry T on $C_0(\Omega)$ with a spectral decomposition

$$T = P_0 - (I - P_0) \quad \textit{for the bicircular case,}$$

$$T = P_0 + \beta P_1 + \beta^2 P_2 \quad \textit{for the tricircular case,}$$

where $\beta = e^{\mathbf{i}\frac{2\pi}{3}}$.

Generalized 4-circular projections on $C_0(\Omega)$

Corollary (D.I., C.-N. Liu, N.-C. Wong)

Let Ω be a connected locally compact Hausdorff space. Let $Tf = u \cdot f \circ \varphi$ be a surjective isometry on $C_0(\Omega)$ with the spectral decomposition

$$T = P_0 + \lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3.$$

Then $\sigma(T) = \{1, \lambda_1, \lambda_2, \lambda_3\}$ can only be one of the following:

$$\{1, -1, i, -i\}, \qquad \{1, -1, \beta, \beta^2\}, \qquad \{1, -1, -\beta, -\beta^2\},$$

$$\{1, -\beta, \beta, \beta^2\}, \qquad \{1, \beta, \beta^2, -\beta^2\}.$$

All above cases can happen. Here $\beta = e^{\mathbf{i}\frac{2\pi}{3}}$.

Generalized 5-circular projections on $C_0(\Omega)$

Corollary (D.I., C.-N. Liu, N.-C. Wong)

Let Ω be a connected locally compact Hausdorff space.
Let $Tf = u \cdot f \circ \varphi$ be a surjective isometry on $C_0(\Omega)$ with the spectral decomposition

$$T = P_0 + \lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3 + \lambda_4 P_4.$$

Then $\sigma(T) = \{1, \lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ can only be one of the following:

$$\{1, \delta, \delta^2, \delta^3, \delta^4\}, \qquad \{1, -1, \beta, -\beta, \beta^2\}, \qquad \{1, -1, \beta, -\beta, -\beta^2\},$$

$$\{1, -1, \beta, \beta^2, -\beta^2\}, \qquad \{1, \beta, -\beta, \beta^2, -\beta^2\}.$$

All above cases can happen. Here, $\beta = e^{\mathbf{i}\frac{2\pi}{3}}$ and $\delta = e^{\mathbf{i}\frac{2\pi}{5}}$.
If T has constant weight function u, then only the primitive (the first) case is allowed.

Non-primitive generalized n-circular projections on $C_0(\Omega)$

Theorem (D.I., C.-N. Liu, N.-C. Wong)

There exists a non-primitive generalized n-circular projection on continuous functions on a connected compact Hausdorff space for each $n \geq 4$.