Some open questions about Kronecker quotients

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The Kronecker product and quotient

- Let A be an $m \times n$ matrix and B be an $s \times t$ matrix over a field $\mathbb{F}$. The Kronecker product $A \otimes B$ is the $ms \times nt$ matrix

$$A \otimes BB = \begin{pmatrix} (A)_{1,1}BB & (A)_{1,2}BB & \cdots & (A)_{1,n}BB \\ (A)_{2,1}BB & (A)_{2,2}BB & \cdots & (A)_{2,n}BB \\ \vdots & \vdots & \ddots & \vdots \\ (A)_{m,1}BB & (A)_{m,2}BB & \cdots & (A)_{m,n}BB \end{pmatrix}.$$

- A right Kronecker quotient $\oslash$ obeys $(A \otimes B) \oslash B = A$ for all matrices A and $B \neq 0$.

Yorick Hardy, *On Kronecker quotients*, Electronic Journal of Linear Algebra **27** (2014), 172–189.

Kronecker quotients

Potential properties

$(A \otimes B)^T = A^T \otimes B^T$	$(M \oslash B)^T = M^T \oslash B^T$	✓
$(A + B) \otimes C = A \otimes C + B \otimes C$	$(M + N) \oslash B = M \oslash B + N \oslash B$	✓
$(kA) \otimes B = k(A \otimes B)$	$(kM) \oslash B = k(M \oslash B)$	✓
$A \otimes (kB) = k(A \otimes B)$	$M \oslash (kB) = \frac{1}{k}(M \oslash B)$	✓
$(A \otimes B) \otimes C = A \otimes (B \otimes C)$	$(M \oslash A) \oslash B = M \oslash (B \otimes A)$	✓
$(M \otimes A)(N \otimes B) = (MN) \otimes (AB)$	$(M \oslash A)(N \oslash B) = (MN) \oslash (AB)$	×
$\operatorname{tr}(A \otimes B) = \operatorname{tr}(A)\operatorname{tr}(B)$	$\operatorname{tr}(M) = \operatorname{tr}(B)\operatorname{tr}(M \oslash B)$	✓*
$\operatorname{rank}(A \otimes B) = \operatorname{rank}(A)\operatorname{rank}(B)$	$\operatorname{rank}(M) = \operatorname{rank}(B)\operatorname{rank}(M \oslash B)$	✓
$\det(A \otimes B) = (\det(B))^m(\det(A))^n$	$\det(M) = (\det(B))^m(\det(M \oslash B))^n$	✓*

Uniform Kronecker quotients

A uniform right Kronecker quotient $\oslash$ obeys

$$(M+N) \oslash B = M \oslash B + N \oslash B$$
$$(cM) \oslash B = c(M \oslash B)$$
$$(A \otimes C) \oslash B = A(C \oslash B)$$

where $c \in \mathbb{F}$, and C and B have the same size.

Partial Frobenius product

Let B be an $s \times t$ matrix and M be a $ms \times nt$ matrix. The (right) *partial Frobenius product* is given by the partial trace

$$(B \circ M) = \mathrm{tr}_2((I_m \otimes B^T)M)$$

or entry wise by

$$(B \circ M)_{u,v} = \sum_{j=1}^{s} \sum_{k=1}^{t} (B)_{j,k} (M)_{(u-1)s+j,(v-1)t+k}$$

and $M \circ B = B \circ M$.

Partial Frobenius product

- If $A \circ B$ is defined then

$$(A \circ B)^T = A^T \circ B^T.$$

- If A is $s \times t$, B is $ms \times nt$ and C is $p \times q$, then

$$(C \otimes B) \circ A = C \otimes (B \circ A).$$

- If A is $msp \times ntq$, B is $s \times t$ and C is $p \times q$, then

$$A \circ (B \otimes C) = (A \circ C) \circ B.$$

- If A is $n \times n$ then $A \circ I_n = \mathrm{tr}(A)$.

Uniform Kronecker quotients

For every uniform Kronecker quotient there exists a map Q on matrices* such that

$$M \oslash B = M \circ Q(B)$$

where

$$B \circ Q(B) = 1.$$

We have

- $(M \oslash B)^T = (M^T \oslash B^T) \iff Q(B^T) = (Q(B))^T$
- $M \oslash (kB) = \frac{1}{k}(M \oslash B) \iff Q(kB) = \frac{1}{k}Q(B)$
- $M \oslash (B \oslash C) = M \oslash (C \otimes B) \iff Q(C \otimes B) = Q(C) \otimes Q(B)$
- $\mathrm{tr}(M) = \mathrm{tr}(B)\,\mathrm{tr}(M \oslash B) \iff Q(B) = (\mathrm{tr}(B))^{-1} I$

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*which preserves matrix size

Uniform Kronecker quotients

For every uniform Kronecker quotient there exists a map Q on matrices[†] such that

$$M \oslash B = M \circ Q(B)$$

where

$$B \circ Q(B) = 1.$$

Open question:

How to characterize a uniform quotient in the non-finite case?

[†] which preserves matrix size

Uniform Kronecker quotients

When $\mathrm{char}(\mathbb{F}) = 0$, a uniform Kronecker quotient exists which satisfies

$$(M \oslash B)^T = M^T \oslash B^T$$
$$(M \oslash A) \oslash B = M \oslash (B \otimes A)$$

$$Q(A) = \frac{A^{\bullet^{-1}}}{\mathrm{nnz}(A)}$$

(entry-wise inverse)

$$Q(A) = \frac{\overline{A}}{\|A\|^2}$$

(complex conjugate)

Paul Leopardi, *A generalized FFT for Clifford algebras*, Bulletin of the Belgian Mathematical Society **11** (2005), 663–688.

Charles F. van Loan and Nikos Pitsianis, *Approximation with Kronecker products*, Linear Algebra for Large Scale and Real-Time Applications (Marc S. Moonen, Gene H. Golub, and Bart L. R. De Moor, eds.), Kluwer Publications, 1993, pp. 293–314.

Uniform Kronecker quotients

Potential properties

When $\mathrm{char}(\mathbb{F}) = 0$, a uniform Kronecker quotient exists which satisfies

$$(M \oslash B)^T = M^T \oslash B^T$$
$$(M \oslash A) \oslash B = M \oslash (B \otimes A)$$

Open question:

Does such a Kronecker quotient exist when $\mathrm{char}(\mathbb{F}) \neq 0$?

Uniform Kronecker quotients

A uniform Kronecker quotient **cannot** satisfy both

$$\operatorname{tr}(M) = \operatorname{tr}(B)\operatorname{tr}(M \oslash B)$$

$$(M \oslash A) \oslash B = M \oslash (B \otimes A)$$

since

$$Q\left(\begin{bmatrix}1 & 1 & 1 & 1\\1 & 1 & 1 & 1\\1 & 1 & 1 & 1\\1 & 1 & 1 & 1\end{bmatrix}\right) = Q\left(\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\right) \otimes Q\left(\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\right) = \frac{1}{4}I_4$$

and

$$Q\left(\begin{bmatrix}1 & 1 & 1 & 1\\1 & 1 & 1 & 1\\1 & 1 & 1 & 1\\1 & 1 & 1 & 1\end{bmatrix}\right) = Q\left(\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right) \otimes Q\left(\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}\right) \neq \frac{1}{4}I_4$$

Uniform Kronecker quotients

A uniform Kronecker quotient **cannot** satisfy both

$$\operatorname{tr}(M) = \operatorname{tr}(B)\operatorname{tr}(M \oslash B)$$
$$(M \oslash A) \oslash B = M \oslash (B \otimes A)$$

Open question:

Consider only square matrices. Does a uniform Kronecker quotient exist which satisfies both properties?

Does a non-uniform Kronecker quotient exist which satisfies both properties?

Kronecker quotients: determinant

Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. Then there exists

$$\oslash : GL^+(\mathbb{F}, mn) \times GL^+(\mathbb{F}, n) \to GL^+(\mathbb{F}, m)$$

such that for all $k \in \mathbb{R}^+$, $A \in GL^+(\mathbb{F}, m)$, $B \in GL^+(\mathbb{F}, n)$ and $M \in GL^+(\mathbb{F}, mn)$

1. $(A \otimes B) \oslash B = A$
2. $\det(M) = (\det(B))^m (\det(M \oslash B))^n$
3. $M \oslash (kB) = \frac{1}{k}(M \oslash B)$

Kronecker differences

The Kronecker sum $A \boxplus B$ of an $m \times m$ matrix A and an $n \times n$ matrix B is given by

$$A \boxplus B = A \otimes I_n + I_m \otimes B.$$

A Kronecker difference $\boxminus$ obeys

$$(A \boxplus B) \boxminus B = A.$$

Each Kronecker quotient induces a Kronecker difference:

$$M \boxminus B = (M - I_m \otimes B) \oslash I_n.$$

We have

$$e^{A \boxplus B} = e^A \otimes e^B.$$

Does a Kronecker quotient exist which satisfies

$$e^{M \boxminus B} = e^M \oslash e^B \,?$$

Thank you.