Kräuter conjecture on permanents is true!

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Joint work with

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$$\det A = \sum_{\sigma \in \mathfrak{S}_n} sgn(\sigma) a_{1\sigma(1)} \cdots a_{n\sigma(n)}$$

and

$$\operatorname{per} A = \sum_{\sigma \in \mathfrak{S}_n} a_{1\sigma(1)} \cdots a_{n\sigma(n)},$$

here $A = (a_{ij}) \in M_n(\mathbb{C})$, $\mathfrak{S}_n$ denotes the set of all permutations of the set $\{1, 2, \ldots, n\}$. The value $sgn(\sigma) \in \{-1, 1\}$ is the signum of the permutation σ.

per is a combinatorial invariant:

$$\operatorname{per}(PAQ) = \operatorname{per} A$$

for all permutation matrices P, Q

Let $A \in M_{k,n}$, $k \leq n$. Then

$$\operatorname{per} A = \sum_{\alpha \in \Lambda_{n,n-k}} \operatorname{per} A(|\alpha),$$

where $\Lambda_{n,r}$ is the set of all subsets consisting of r distinct elements of the set $\{1, \ldots, n\}$ and $A(|\alpha)$ is the matrix obtained from A by deleting rows with numbers from α.

Some applications of permanent

Derangements problem

In how many ways can a dance be arranged for n married couples, so that no husband dances with his own wife?

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$$D_n = \operatorname{per}\begin{pmatrix} 0 & 1 & \dots & 1 \\ 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \dots & 1 & 0 \end{pmatrix} = \operatorname{per}(J_n - I_n) = n! \cdot \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

Ménage problem

In how many ways can n married couples be placed at a round table, so that men and women sit in alternate places and no husband sit on either side of his wife?

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$$U_n = \operatorname{per}\begin{pmatrix} 0 & 0 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 0 & \ddots & \vdots & \vdots \\ 1 & 1 & 0 & \ddots & 1 & 1 \\ \vdots & \vdots & \ddots & \ddots & \ddots & 1 \\ 1 & 1 & \cdots & 1 & 0 & 0 \\ 0 & 1 & 1 & \cdots & 1 & 0 \end{pmatrix} = \operatorname{per}(J_n - I_n - P_n)$$

P_n is a permutation matrix of $(1,2)(2,3)\cdots(n-1,n)(n,1)$.

In how many ways can n married couples be placed at a round table, so that men and women sit in alternate places and no husband sit on either side of his wife?

Sequence number $A059375$ in on-line encyclopedia of integer sequences

The first terms:

12, 96, 3120, 115200, 5836320, 382072320, 31488549120, ...

Formulated in 1891 by Édouard Lucas and independently, a few years earlier, by Peter Guthrie Tait in connection with knot theory

Touchard (1934) derived the formula

$$U_n = 2 \cdot n! \sum_{k=0}^{n} (-1)^k \frac{2n}{2n-k} \binom{2n-k}{k} (n-k)!$$

Latin squares

S is a set, $|S| = n$ usually, $S = \{1, 2, \ldots, n\}$

A Latin rectangle on S is an $r \times s$ matrix A with $a_{ij} \in S$, $a_{ij} \neq a_{il}$, and $a_{ij} \neq a_{kj}$.

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$n \times n$ Latin rectangle is a Latin square.

Problems: 1. To find the number $L(n, n)$ of Latin squares on S

2. To find the number $L(r, n)$ of $r \times n$ Latin rectangles on S

Known facts

1. $L(1,n) = 1$

2. $L(2,n) = n! \cdot D_n$

3. $L(3,n) = n! \cdot \sum_{k=0}^{\lfloor n/2 \rfloor} C_n^k D_{n-k} D_k U_{n-2k}$

Λ_n^k is the set of $(0{,}1)$-matrices with k 1 in each row and column.

$m(k,n)$ and $M(k,n)$ are lower and upper bounds for permanent on Λ_n^k.

Then

$$n! D_n \prod_{t=2}^{r-1} m(n-t,n) \le L(r,n) \le n! D_n \prod_{t=2}^{r-1} M(n-t,n)$$

Applications of permanent:

Counting function for combinatorial problems

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DNA identification

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Makes everybody happy

	det	per
Geometry	Oriented volume	Combinatorial geometry
Algebra	$\lambda_1 \cdots \lambda_n$	Bounds
Complexity	$O(n^3)$	$\sim (n-1)\cdot(2^n-1)$

Ryser's formula

$$\mathrm{per}(A) = \sum_{t=0}^{n-1} (-1)^t \sum_{X \in L_{n-t}} \prod_{i=1}^{n} r_i(X)$$

$r_i(X) = \sum_{j=1}^{t} x_{ij}$ — i-th row sum

L_{n-t} — the set of all $n \times (n-t)$ submatrices of A

IMPORTANT MATRIX CLASSES:

- $(0, 1)$ matrices

- $(-1, 1)$ matrices

Applications of ± 1-matrices

Theorem [Frobenius, 1896]

$$T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$$

— linear, bijective

$$\det(T(A)) = \det A \qquad \forall A \in M_n(\mathbb{C})$$

$$\Downarrow$$

$$\exists P, Q \in GL_n(\mathbb{C}), \det(PQ) = 1 :$$

$T(A) = PAQ \quad \forall A \in M_n(\mathbb{C})$ or $T(A) = PA^tQ \quad \forall A \in M_n(\mathbb{C})$

Theorem [Marcus, May] Linear transformation T is permanent preserver

iff

$T(A) = P_1 D_1 A D_2 P_2 \quad \forall A \in M_n(\mathbb{F})$, or

$T(A) = P_1 D_1 A^t D_2 P_2 \quad \forall A \in M_n(\mathbb{F})$

here D_i are invertible diagonal matrices, $i = 1, 2$, $\mathrm{per}(D_1 D_2) = 1$,

P_i are permutation matrices, $i = 1, 2$

Problem. Under what conditions does there exists a transformation $\Phi :$ $M_n(\mathbb{F}) \to M_m(\mathbb{F})$ satisfying

$$\operatorname{per} A = \det \Phi(A)?$$

Here a transformation Φ on $M_n(\mathbb{F})$ is called a converter.

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Proof: based on linear algebra.

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Proof: based on linear algebra.

Theorem (J. von zur Gathen, 1987). *Let* $\mathbb{F}$ *be infinite,* $char(\mathbb{F}) \neq 2$. *There is no bijective affine transformation* $\Phi : M_n(\mathbb{F}) \to M_n(\mathbb{F}), n > 2$ *satisfying* $\operatorname{per} A = \det \Phi(A)$ $\forall$ $A \in M_n(\mathbb{F})$.

Proof: based on algebraic geomery.

Polya, 1913 observed:

$n = 2$:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \operatorname{per} \begin{pmatrix} a & b \\ -c & d \end{pmatrix}$$

Problem [Polya, 1913]. Does $\exists$ a uniform way of affixing $\pm$ to the entries of $A = (a_{ij}) \in M_n(\mathbb{F})$: $\text{per}(a_{ij}) = \det(\pm a_{ij})$?

Equivalently: Does $\exists$ $X \in M_n(\pm 1)$: $\text{per}\, A = \det(A \circ X)$ $\forall$ $A \in M_n(\mathbb{F})$?

$A \circ X = (a_{ij} x_{ij})$ is Hadamard (Schur) product of $A = (a_{ij})$ and $X = (x_{ij})$

$$n = 2: \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ -c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

Szegö, 1914. $n > 2$: NO.

Why NOT ?

$n = 3$: consider $J_3 = \left(\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{smallmatrix}\right)$.

Then $\mathrm{per}\, J_3 = 6$ but

$$\det \begin{pmatrix} \pm 1 & \pm 1 & \pm 1 \\ \pm 1 & \pm 1 & \pm 1 \\ \pm 1 & \pm 1 & \pm 1 \end{pmatrix} < 6$$

since each -1 is in two summands, so all 6 summands can not be positive.

What about SUBSETS of M_n?

Sometimes the conversion is possible:

1. $$\begin{pmatrix} a & b & 0 \\ c & d & e \\ f & g & h \end{pmatrix} \mapsto \begin{pmatrix} a & b & 0 \\ -c & d & e \\ f & -g & h \end{pmatrix}$$

2. A: $a_{ij} = 0$ if $j - i \geq 2$ (Hessenberg matrices)

$$A \mapsto \tilde{A} = (\tilde{a_{ij}}):\ \tilde{a_{ij}} = \begin{cases} -a_{ij}, & \text{if } j - i = 1 \\ a_{ij}, & \text{otherwise} \end{cases}$$

2. Ice cream market

α is a shift parameter of demand curve, for example, taste.

$S(p)$ is the number x of ice creams what a factory would produce for the price p.

$D(p, \alpha)$ is the demand for the price p and taste α.

Then equilibrium equations for (p_α, x_α) are $S(p) = x = D(p, \alpha) \qquad (1)$.

Let us check that if α increases then so do p_α and x_α.

(1) $\Leftrightarrow$ to the system $\begin{cases} S(p) - x = 0 \\ D(p, \alpha) - x = 0 \end{cases}$

$$\text{Hence, } \begin{cases} \dfrac{\partial S}{\partial p}\dfrac{\partial p}{\partial \alpha} - \dfrac{\partial x}{\partial \alpha} = 0 \\ \dfrac{\partial D}{\partial p}\dfrac{\partial p}{\partial \alpha} + \dfrac{\partial D}{\partial \alpha} - \dfrac{\partial x}{\partial \alpha} = 0 \end{cases} \Leftrightarrow \begin{pmatrix} \dfrac{\partial S}{\partial p} & -1 \\ \dfrac{\partial D}{\partial p} & -1 \end{pmatrix} \begin{pmatrix} \dfrac{\partial p}{\partial \alpha} \\ \dfrac{\partial x}{\partial \alpha} \end{pmatrix} = \begin{pmatrix} 0 \\ -\dfrac{\partial D}{\partial \alpha} \end{pmatrix}$$

Economics $\Rightarrow \dfrac{\partial S}{\partial p} > 0, \dfrac{\partial D}{\partial p} < 0, \dfrac{\partial D}{\partial \alpha} > 0$, so

$$\begin{pmatrix} + & - \\ - & - \end{pmatrix} \begin{pmatrix} \dfrac{\partial p}{\partial \alpha} \\ \dfrac{\partial x}{\partial \alpha} \end{pmatrix} = \begin{pmatrix} 0 \\ - \end{pmatrix}$$

$$\begin{pmatrix} + & - \\ \\ - & - \end{pmatrix} \begin{pmatrix} \frac{\partial p}{\partial \alpha} \\ \\ \frac{\partial x}{\partial \alpha} \end{pmatrix} = \begin{pmatrix} 0 \\ \\ - \end{pmatrix}$$

$\Rightarrow$ always $\exists$ solution and (Cramer rule)

$$\frac{\partial p}{\partial \alpha} > 0, \frac{\partial x}{\partial \alpha} > 0$$

Indeed,

$$\Delta = \left| \begin{smallmatrix} + & - \\ - & - \end{smallmatrix} \right| = (-) + (-) < 0,$$

$$\Delta_1 = \left| \begin{smallmatrix} 0 & - \\ - & - \end{smallmatrix} \right| = -(-) \cdot (-) < 0, \quad \Delta_2 = \left| \begin{smallmatrix} + & 0 \\ - & - \end{smallmatrix} \right| = (+) \cdot (-) < 0$$

So, p_α, x_α are indeed increasing functions!

3. Hadamard matrices

$(-1, 1)$-matrix with pair-wise orthogonal rows.

- Hadamard matrix has size 1, 2 or $4k$

- Hadamard conjecture: $\forall\ k\ \exists$ a Hadamard matrix of size $4k$

Properties of permanents of $(0,1)$ and $(-1,1)$ matrices:

- Permanents of $(0,1)$ matrices can not decrease if 0 is changed by 1

- Permanents of $(-1,1)$ matrices can decrease if -1 is changed by 1

$$\operatorname{per}\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} = 2 > 0 = \operatorname{per}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$$

- Permanents of $(-1,1)$ matrices do not depend on the number of -1

in a matrix

- Max value $n!$ is attained only on singular matrices

Examples:

$J = \begin{pmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{pmatrix}$. Then $\operatorname{per} J = n!$.

If n is even then maxper $= n!$ if all the entries are (-1) or exactly half of the entries are (-1), see $\begin{pmatrix} -1 & \dots & -1 \\ \vdots & \ddots & \vdots \\ -1 & \dots & -1 \end{pmatrix}$ or chess-matrix $\begin{pmatrix} 1 & -1 & \dots & 1 & -1 \\ -1 & 1 & \dots & -1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & -1 & \dots & 1 & -1 \\ -1 & 1 & \dots & -1 & 1 \end{pmatrix}$

Theorem [Kräuter]

- $\operatorname{per} A \leq n!$
- $\operatorname{per} A = n!$ iff $A \sim J_n$

$\sim$ is a composition of transposition, row or column permutations, multiplication of rows/columns by (-1).

- $A \not\sim J_n \Rightarrow \operatorname{per} A \leq (n-2)(n-1)!$

In general, it is difficult to compute permanent

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$$n = 6: \quad \mathrm{per}\begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \dots & 1 & 1 \end{pmatrix} = 6! = 720, \qquad \mathrm{per}\begin{pmatrix} -1 & 1 & \dots & 1 \\ 1 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \dots & 1 & -1 \end{pmatrix} = 112$$

Problem [Wang, Israel J. Math., **18**, 1974]

Let $A \in M_n(\pm 1)$ be a nonsingular matrix. Is there a decent upper bound for $|\operatorname{per} A|$?

Let $D_{(n,k,l)} = (d_{ij}) \in M_{k,n}(\pm 1)$, $0 \leq l \leq k \leq n$,

$$d_{ij} = \begin{cases} -1, i = j \text{ and } j \in \{1, \ldots, l\} \\ \\ 1, \text{otherwise.} \end{cases}$$

If $n = k$ we write $D_{(n,n,l)} = D_{(n,l)}$.

Conjecture [Kräuter, 1985] Let $A \in M_n(\pm 1)$, $n \geq 5$, $\operatorname{rk} A = r + 1$. Then

$$|\operatorname{per} A| \leq \operatorname{per} D_{(n,r)}.$$

The equality holds iff $A \sim D_{(n,r)}$,

$\sim$ is transposition $\circ$ row or column perm. $\circ$ mult. by (-1).

In particular,

if A is invertible then

$$\operatorname{per} A \leq \operatorname{per} \begin{pmatrix} -1 & 1 & \dots & 1 & 1 \\ 1 & -1 & & \vdots & 1 \\ \vdots & & \ddots & 1 & \vdots \\ 1 & \dots & 1 & -1 & 1 \\ 1 & \dots & 1 & 1 & 1 \end{pmatrix} .$$

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For example,

$$\operatorname{per} \begin{pmatrix} -1 & 1 & \dots & 1 & 1 \\ 1 & -1 & & \vdots & 1 \\ \vdots & & \ddots & 1 & \vdots \\ 1 & \dots & 1 & -1 & 1 \\ 1 & \dots & 1 & 1 & -1 \end{pmatrix} \leq \operatorname{per} \begin{pmatrix} -1 & 1 & \dots & 1 & 1 \\ 1 & -1 & & \vdots & 1 \\ \vdots & & \ddots & 1 & \vdots \\ 1 & \dots & 1 & -1 & 1 \\ 1 & \dots & 1 & 1 & 1 \end{pmatrix}$$

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if $n \geq 5$!

$n = 4$: Exceptional case

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$$\operatorname{per} D_{(4,3)} = \operatorname{per}\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = 4 < 8 = \operatorname{per}\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix} = \operatorname{per} D_{(4,4)}$$

Some computations

$$\begin{cases} \operatorname{per} D_{(4,0)} = \operatorname{per}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = 24 \\ \operatorname{per} D_{(4,1)} = \operatorname{per}\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = 12 \\ \operatorname{per} D_{(4,2)} = \operatorname{per}\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = 8 \\ \operatorname{per} D_{(4,3)} = \operatorname{per}\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = 4 \\ \operatorname{per} D_{(4,4)} = \operatorname{per}\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix} = 8 \end{cases}$$

Some computations

$$
\begin{cases}
\operatorname{per} D_{(4,0)} = \operatorname{per}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = 24 \\
\operatorname{per} D_{(4,1)} = \operatorname{per}\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = 12 \\
\operatorname{per} D_{(4,2)} = \operatorname{per}\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = 8 \\
\operatorname{per} D_{(4,3)} = \operatorname{per}\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = 4 \\
\operatorname{per} D_{(4,4)} = \operatorname{per}\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix} = 8
\end{cases}
$$

Almost more -1 on diagonal — bigger rank — smaller permanent

$n = 4$: Exceptional case

$$\operatorname{per} D_{(4,3)} = \operatorname{per}\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = 4 < 8 = \operatorname{per}\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix} = \operatorname{per} D_{(4,4)}$$

Since $\operatorname{per} D_{(3,3)} = -2$ Laplace expansion of $\operatorname{per} D_{(4,3)}$ by the last row contains summand < 0.

Hence $\operatorname{per} D_{(4,3)} = 4 < 8 = \operatorname{per} D_{(4,4)}$.

$$
\begin{cases}
\operatorname{per} D_{(6,6)} = \operatorname{per}\begin{pmatrix} -1 & 1 & \dots & 1 \\ 1 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \dots & 1 & -1 \end{pmatrix} = 112 \\
\operatorname{per} D_{(5,5)} = 8 \\
\operatorname{per} D_{(4,4)} = 8 \\
\operatorname{per} D_{(3,3)} = \operatorname{per}\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} = -2 \\
\operatorname{per} D_{(2,2)} = \operatorname{per}\left(\begin{smallmatrix} -1 & 1 \\ 1 & -1 \end{smallmatrix}\right) = 2
\end{cases}
$$

If $n > 4$ $\nexists$ summands < 0 in expansion of $\operatorname{per} D_{(n,n-1)}$ by the last row.

Extended Conjecture [BG, 2017]

Let $A \in M_n(\pm 1)$, $\operatorname{rk} A = r + 1$. Then

$$|\operatorname{per} A| \leq \operatorname{per} D_{(n,r)}$$

$\forall\ n,\ r$ except 4×4 invertible matrices.

The equality holds iff $A \sim D_{(n,r)}$.

If $n = 4$ then there is an exception $D_{(4,4)}$ unique up to $\sim$.

Invertible matrices of small size

- Let $A \in M_2(\pm 1)$ be invertible. Then $|\operatorname{per} A| = \operatorname{per} D_{(2,1)} = 0$.

- Let $A \in M_3(\pm 1)$ and $\operatorname{rk} A = k$. Then $|\operatorname{per} A| = \operatorname{per} D_{(3,k-1)}$ and

$$A \sim D_{(3,k-1)} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

- Let $A \in GL_4(\pm 1)$ be s.t. $|\operatorname{per} A| = \max_{C \in GL_4(\pm 1)} |\operatorname{per} C| = 8$.

Then

$$A \sim D_{(4,4)} = \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}$$

If for $B \in GL_4(\pm 1)$ it holds $|\operatorname{per} B| < 8$, then

$$|\operatorname{per} B| \leq \operatorname{per} D_{(4,3)} = \operatorname{per} \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = 4$$

Hint for the proof:

Lemma 1 Let $A \in M_4(\pm 1)$. Then $\operatorname{per} A$ is divisible by 4.

Properties of $\operatorname{per} D_{(n,k,l)}$

Lemma 2 [Kräuter, Seifter] Let $n \geq 5$. Then

1. $\operatorname{per} D_{(n,n)} > 0$.

2. $\operatorname{per} D_{(n,k)} > 0$ for any possible value of k.

3. For any $0 < k \leq n$ we have $\operatorname{per} D_{(n,k-1)} > \operatorname{per} D_{(n,k)}$.

Corollary Let $n \geq 5$. Then for any l, k such that $0 \leq l < k \leq n$

$$\operatorname{per} D_{(n,l)} > \operatorname{per} D_{(n,k)} \text{ and } \operatorname{rk} D_{(n,l)} < \operatorname{rk} D_{(n,k)}$$

Steps of the proof

1. Case $n = 4$. $D_{(4,4)}$ is the unique except case

2. Prove $|\operatorname{per} A| \leq \operatorname{per} D_{(n,n-1)}$ for $A \in GL_n(\pm 1), n = 5, 6$.

3. Let $A \in GL_n(\pm 1)$, $n \geq 7$. Assume Conjecture holds true $\forall$ matrices of the order $m < n$. Then $|\operatorname{per} A| \leq \operatorname{per} D_{(n,n-1)}$. If $|\operatorname{per} A| = \operatorname{per} D_{(n,n-1)}$ then $A \sim D_{(n,n-1)}$.

4. Let $A \in M_n(\pm 1)$, $\operatorname{rk} A = k < n$. Assume Conjecture is proved $\forall$ $B \in M_l(\pm 1)$, $l \leq k$. Then $|\operatorname{per} A| \leq \operatorname{per} D_{(n,k-1)}$. If $|\operatorname{per} A| = \operatorname{per} D_{(n,k-1)}$, then $A \sim D_{(n,k-1)}$.

Items 1, 2: “by hands”.

Item 3:

Definition $(-1,1)$-matrix A satisfies condition $\mathfrak{A}$ if the following is true:

- all entries of the first row of A are $+1$,
- the second row of A contains ≥ 3 entries $+1$ and ≥ 3 entries -1.

Key Lemma: Let $A \in GL_n(\pm 1)$, $n \geq 6$. Then

$$A \sim \left[\begin{array}{lr} D_{(n,n-1)} & (1) \\ D_{(n,n)} & (2) \\ P_1 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & 1 & 1 & -1 \end{pmatrix} & (3) \\ P_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 & 1 & -1 \end{pmatrix} & (4) \\ X \in \mathfrak{A} & (5) \end{array}\right.$$

Here $\operatorname{per} P_1 = \operatorname{per} P_2 = 16$. Case (1) is extremal, in the cases (2) – (4) permanent is not maximal, so inequality holds. Case (5) — induction.

Several identities

For any $1 \le k \le n$

$$\operatorname{per} D_{(n,k-1)} = \operatorname{per} D_{(n,k)} + 2 \operatorname{per} D_{(n-1,k-1)}.$$

For $n \ge 5$

$$\operatorname{per} D_{(n,n)} = (n-2) \operatorname{per} D_{(n-1,n-1)} + (2n-2) \operatorname{per} D_{(n-2,n-2)}.$$

For $n \ge 6$ $\operatorname{per} D_{(n,n-1)} =$

$$= 2 \operatorname{per} D_{(n-2,n-3)} + (n^2-7n+12) \operatorname{per} D_{(n-2,n-5)} + 2(n-3) \operatorname{per} D_{(n-2,n-4)}.$$

$n \geq 7$ then

$$|\operatorname{per} A| \leq \sum_{\alpha \in \Lambda_{n,2}} |\operatorname{per} A[1,2|\alpha]| \cdot |\operatorname{per} A(1,2|\alpha)| \leq$$

$$(|\operatorname{per} A(1,2|\alpha)| \leq \operatorname{per} D_{(n-2,n-5)})$$

$$\leq (\tfrac{1}{2}n(n-1) - 3(n-3)) \cdot 2 \cdot \operatorname{per} D_{(n-2,n-5)} =$$

$$= (n^2 - 7n + 18) \operatorname{per} D_{(n-2,n-5)}.$$

Then $\operatorname{per} D_{(n,n-1)} - |\operatorname{per} A| \geq$

$$2 \operatorname{per} D_{(n-2,n-3)} + 2(n-3) \operatorname{per} D_{(n-2,n-4)} - 6 \operatorname{per} D_{(n-2,n-5)}$$

$$\operatorname{per} D_{(n,n-1)} - |\operatorname{per} A| \geq (2n^3 - 16n^2 + 10n + 44) \operatorname{per} D_{(n-4,n-4)} +$$

$$+(4n^3 - 36n^2 + 48n + 80) \operatorname{per} D_{(n-5,n-5)} = F(n).$$

$f(n) = 2n^3 - 16n^2 + 10n + 44$. Then $f' = 6n^2 - 32n + 10$ has the roots

$\frac{1}{3}$ and 5, $f(7) = 16 \Rightarrow f(n) > 0 \;\forall\; n \geq 7$.

$g(n) = 4n^3 - 36n^2 + 48n + 80$. Then $g' = 12n^2 - 72n + 48$ has the roots

$3 \pm \sqrt{5}$, $g(7) = 24$ and $3 \pm \sqrt{5} < 7 \Rightarrow g(n) > 0 \;\forall\; n \geq 7$.

$n \geq 9 \;\Rightarrow \operatorname{per} D_{(n-4,n-4)} > 0, \operatorname{per} D_{(n-5,n-5)} > 0 \Rightarrow F(n) > 0 \;\forall\; n \geq 9.$

Also $F(8) = 576 > 0$, $F(7) = 17 > 0 \Rightarrow$

$$\operatorname{per} D_{(n,n-1)} - |\operatorname{per} A| \geq F(n) > 0 \ \forall \ n \geq 7$$

Since $F(n) > 0$ the bound is strict and $A \not\sim D_{(n,n-1)}$, which proves the characterization part

$\operatorname{per} D_{(n,n-1)} - |\operatorname{per} A| \geq F(n) > 0 \ \forall\ n \geq 7$

Since $F(n) > 0$ the bound is strict and $A \not\sim D_{(n,n-1)}$, which proves the characterization part

$+$ apply majorization.

Theorem [BG, 2017]

Let $A \in M_n(\pm 1)$, $\operatorname{rk} A = r+1$. Then

$$|\operatorname{per} A| \le \operatorname{per} D_{(n,r)}.$$

$\forall\, n$, r except 4×4 invertible matrices.

The equality holds iff $A \sim D_{(n,r)}$.

If $n = 4$ then there is an exception $D_{(4,4)}$, unique up to $\sim$.

THANK YOU