Extremal non-convertible fully indecomposable (0,1)-matrices

Mikhail V. Budrevich
(Lomonosov Moscow State University)
(joint work with Gregor Dolinar, Alexander E. Guterman and Bojan Kuzma)

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Permanent and determinant functions

Definition

Let $A = (a_{ij})$ be a square $(0,1)$-matrix of ordere n and S_n is a symmetric group on n elements, then

$$\det A = \sum_{\sigma in S_n} \mathrm{sign}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}$$

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Example (Applications of permanent)

1. *Number of perfect matching in bipartite graph*
2. *Number of domino tiling's*
3. *Number of derangements of order n*

Sign conversion

Example (Pólya, 1913)

Let $A \in M_2$ *and mapping* ϕ *deffined by*

$$\phi : \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \rightarrow \begin{pmatrix} a_{11} & a_{12} \\ -a_{21} & a_{22} \end{pmatrix}$$

The equality $\operatorname{per} A = \det \phi(A)$ *holds.*

Definition

The matrix $A \in M_n$ *is (sign) convertible if there is* $(1, -1)$*-matrix* $X = X(A) \in M_n$ *such that the following equation holds*

$$\operatorname{per} A = \det (X \circ A)$$

Sign conversion for (0,1)-matrices

Example (Pólya, Szego)

Matrix $J_3 \in M_n(0,1)$ *is non-convertible*

$$J_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Theorem (Gibson)

Let $A \in M_n(0,1)$ *and* $\mathrm{per}\, A > 0$. *If* A *is convertible then* $\nu(A) \le \Omega_n = \frac{n^2+3n-2}{2}$. *If* $\nu(A) = \Omega_n$ *then there exist permutation matrices* $P,\ Q$ *such that* $PAQ = G_n = (g_{ij})$, *where*

$$\begin{cases} g_{ij} = 1; \ \text{if } j \le i+1 \\ g_{ij} = 0; \ \text{otherwise.} \end{cases}$$

Bounds of conversion

Definition

Number Ω_n such that for every $(0,1)$-matrix A with $\operatorname{per} A > 0$ and $\nu(A) > \Omega_n$ follows that A is non-convertible is called upper bound of conversion.

Definition

Number ω_n such that for every $(0,1)$-matrix A with $\nu(A) < \omega_n$ it follows that A is convertible is called lower bound of conversion.

Theorem (Dolinar, Guterman, Kuzma)

Let $A \in M_n(0,1)$ and $\nu(A) < n-6$. Then A is convertible.

Sign non-singular matrices

Definition

Matrix $A \in M_n(\mathbb{R})$ is sign non-singular if every matrix with the same position of zeros, positive and negative elements is non-singular.

Theorem (Brualdy, Ryser)

Matrix $A \in M_n(0,1)$ is convertible iff there is sing-nonsingular matrix S with zero elements on the same positions as in matrix A.

Example (SNS-matrices)

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

Graphs and sign conversion

Definition

Let G — bipartite oriented graph. If every cycle $C \subset G$ such that $G \setminus C$ has perfect matching is oddly oriented then orientation of G is called Pfaffian.

Theorem (Little)

Bipartite graph G admits Pfaffian orientation iff incidence matrix A is convertible.

Theorem (Valiant)

Computing permanent of $A \in M_n(0,1)$ is $\#P$-complete problem.

Indecomposable matrices

Definition

Matrix A is called decomposable if there exists a permutation matrix P such that

$$A = P^t \begin{pmatrix} B & 0 \\ C & D \end{pmatrix} P$$

where $B,\ D$ are square matrices of the sizes $k > 0$ and $l > 0$ correspondingly, and C is a certain matrix of appropriate size.

Example

Matrix A is decomposable and matrix B is indecomposable:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Fully indecomposable matrices

Definition

Matrix A is called partially decomposable if there exists two permutation matrices P, Q such that

$$A = Q \begin{pmatrix} B & 0 \\ C & D \end{pmatrix} P$$

where $B,\ D$ are square matrices of the sizes $k > 0$ and $l > 0$ correspondingly, and C is a certain matrix of appropriate size. If matrix A is not partially decomposable then it is fully indecomposable.

Example

Matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ is indecomposable but partially decomposable.

Lower bound for (fully) indecomposable matrix

Example

Matrix A is indecomposable and non-convertible.

$$\begin{pmatrix} 0 & I_{n-3} & 0 & 0 \\ 1 & \dots & 1 & 1 \\ 1 & \dots & 1 & 1 \\ 1 & \dots & 1 & 1 \end{pmatrix}$$

Theorem

Let $A \in M_n(0,1)$ be fully indecomposable. If $\nu(A) < 2n+2$, then A is convertible.

Convolution operation

Definition

Let $A \in M_n(0,1)$ and let the 1'st row of A have exactly two non-zero entries a_{11}, a_{12}. Then the convolution of A in the 1'st row is the following matrix $S_1(A) \in M_n(0,1)$

$$S_1(A) = \begin{pmatrix} \max(a_{21}, a_{22}) & a_{23} & \dots & a_{2n} \\ \max(a_{31}, a_{32}) & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots \\ \max(a_{n1}, a_{n2}) & a_{n3} & \dots & a_{nn} \end{pmatrix}$$

Convolution of fully indecomposable matrix

Theorem

Let $A \in M_n(0,1)$. Let the first row of A have exactly two non-zero entries $a_{11},\ a_{12}$, and let $S_1(A)$ be the convolution of A. Then A is convertible if and only if $S_1(A)$ is convertible.

Theorem

Let $A \in M_n(0,1)$. Let the first row of A have exactly two non-zero entries $a_{11},\ a_{12}$, and let $S_1(A)$ be the convolution of A. Then A is fully indecomposable if and only if $S_1(A)$ is fully indecomposable.

Bipartite graphs and $(0, 1)$-matrices

Definition

Let $G = (V_1, V_2, E)$ be bipartite graph with $|V_1| = |V_2| = n$. We say that $A \in M_n(0, 1)$ is incidence matrix of G if

$$a_{ij} = \begin{cases} 1, & \text{iff } (v_1^i, v_2^j) \in E \\ 0, & \text{otherwise.} \end{cases}$$

Remark

For any matrix $A \in M_n(0, 1)$ we can construct graph denoted by $G(A)$ such that A is incidence matrix of $G(A)$.

Convolution and bipartite graphs

Theorem

Let $A \in M_n(0,1)$. Let the first row of A have exactly two non-zero entries $a_{11},\ a_{12}$, and let $S_1(A)$ be the convolution of A. Then $G(S_1(A))$ is constructed from $G(A) = (V_1, V_2, E)$ by the following steps:

1. *Vertices v_1^1 and v_2^1 are removed.*
2. *Edges of the form (v_1^1, x) are replaced by edges of the form (v_1^2, x).*
3. *If there were edges (v_1^1, x) and (v_1^2, x) in E, then they are merged (replaced by (v_1^2, x)).*

Extremal case

Lemma

Let $A \in M_n(0,1)$ be non-convertible fully indecomposable with $2n+3$ non-zero elements. Let the first row of A have exactly two non-zero entries $a_{11},\ a_{12}$, and let $S_1(A)$ be the convolution of A. Then $S_1(A)$ has exactly $2(n-1)+3$ non-zero elements.

Theorem

Let $A \in M_n(0,1)$ be non-convertible fully indecomposable with $2n+3$ non-zero elements. Let the first row of A have exactly two non-zero entries $a_{11},\ a_{12}$, and let $S_1(A)$ be the convolution of A. Then there are no edges in $G(S_1(A))$ that were obtained as merge of edges in G(A).

Description of extremal case (graphs)

Theorem

Let $A \in M_n(0,1)$ be non-convertible fully indecomposable with $2n+3$ non-zero elements. Then up to renumbering of vertices graph $G(A)$ is equal to the graph G such that:

1. *For vertices v_i^j, where $i \in \{1,\ 2\}$ and $j \in \{1,\ 2,\ 3\}$, there are 3 out going edges, and there are two outgoing edges for all other vertices*
2. *For any v_1^j and v_2^k, where $j, k \in \{1,\ 2,\ 3\}$, there is unique chain $(v_1^j, x), (x, y), \dots, (z, v_2^k)$ and degree in any intermediate vertex is 2.*
3. *There are no other vertices or edges in G.*

Description of extremal case (equivalence)

Theorem

Any non-convertible fully indecomposable matrix of order n with $2n+3$ non-zero elements up to permutation of rows and collumns is described by matrix $C \in M_3(\mathbb{Z}_+)$ such that sum of elements of C is $n-3$.

References

1. Budrevich, M., Dolinar, G., Guterman, A., Kuzma, B., Lower bounds for Pólya's problem on permanent, (2016) International Journal of Algebra and Computation, **26** (6), pp. 1237-1255.

Thank you!

M.V.Budrevich
mbudrevich@yandex.ru