Indecomposable Matrices Defining Plane Cubics

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Kippenhahn's conjecture on Hermitian pencils (1951)

Let H, K be $n \times n$ complex Hermitian matrices and $F \in \mathbb{C}[x, y, z]$ a homogeneous polynomial such that

$$\det(x\,H + y\,K - z\,Id) = F(x, y, z).$$

When $F(x, y, z)$ has a repeated factor, then H and K are simultaneously unitarily similar to direct sums: there exists an unitary matrix U and matrices $H_i, K_i \in M_{n_i}(\mathbb{C})$ for some $1 \leq n_i < n$ such that

$$UHU^* = \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} \text{ and } UKU^* = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}.$$

By Burnside's theorem on matrix algebras, this is equivalent to H and K not generating the whole $M_n(\mathbb{C})$.

Known results

- Shapiro (1982) showed that the conjecture holds for $n \leq 5$ and for $n = 6$ in the case that the minimal polynomial is cubic.
- Waterhouse (1984) presented a pair of 6×6 matrices that generate $M_6(\mathbb{C})$ such that $\det(xH + yK - zId)$ has repeated linear factors, thus disproving the general form of Kippenhahn's conjecture for $n = 6$.
- Li and Spitkovsky (1998) constructed another class of counterexamples for $n = 6$.
- Laffey (1983) constructed a counterexample for $n = 8$ with quartic minimal polynomial.

The conjecture of Kippenhahn is true for $n = 6$ with cubic minimal polynomial.

Let H, K be 6×6 complex Hermitian matrices and F a homogeneous polynomial defining a smooth cubic, such that

$$\det(xH + yK - zId) = F(x, y, z)^2.$$

Then H and K are simultaneously unitarily similar to direct sums. This means that there exists an unitary matrix U and matrices $H_1, H_2, K_1, K_2 \in M_3(\mathbb{C})$ such that

$$UHU^* = \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} \text{ and } UKU^* = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}.$$

Weierstrass cubic

Let C be an irreducible curve in $\mathbb{CP}^2$ defined by a polynomial $F(x, y, z)$ of degree 3. Every smooth cubic can be brought by a change of coordinates

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto P \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \text{ for some } P \in \mathrm{GL}_3(\mathbb{C})$$

into a Weierstrass form

$$F(x, y, z) = yz^2 - x(x - y)(x - \lambda y) = 0,$$

or equivalently

$$F(x, y, z) = -yz^2 + x^3 + \alpha xy^2 + \beta y^3 = 0,$$

for some $\lambda \neq 0, 1$ and $\alpha, \beta \in \mathbb{C}$.

Determinantal representations of cubics

Consider the following question. For given C find a linear matrix

$$A(x, y, z) = x\,A_x + y\,A_y + z\,A_z$$

such that

$$\det A(x, y, z) = c\,F(x, y, z)^r,$$

where $A_x, A_y, A_z \in \mathrm{M}_{3r}$ and $0 \neq c \in \mathbb{C}$. Here M_{3r} is the algebra of all $3r \times 3r$ matrices over $\mathbb{C}$.

We call A determinantal representation of C of order r.

Determinantal representation A is definite if $A(x_0, y_0, z_0)$ is definite at some point (x_0, y_0, z_0).

Equivalent determinantal representations

Two determinantal representations A and A' are equivalent if there exist $X, Y \in \mathrm{GL}_{3r}(\mathbb{C})$ such that

$$A' = XAY.$$

We study:

- self-adjoint representations $A = A^*$ modulo unitary equivalence $Y = X^*$,
- skew-symmetric representations $A = -A^t$ under $Y = X^t$ equivalence.

Obviously, equivalent determinantal representations define the same curve.

Cokernels are vector bundles of rank r

Theorem

Consider a linear matrix $A = xA_x + yA_y + zA_z$ with $\det A = F(x, y, z)^r$. When F defines a smooth curve C, the cokernel of A is a vector bundle of rank r on C.

This follows from Beauville (2000) using arithmetically Cohen-Macaulay sheaves, or Eisenbud (1980) and Backelin, Herzog, Sanders (2006) using purely algebraic methods for matrix factorizations of polynomials.

The conjecture of Kippenhahn is true for $n = 6$ with cubic minimal polynomial.

Let H, K be 6×6 complex Hermitian matrices and F a homogeneous polynomial defining a smooth cubic, such that

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Then H and K are simultaneously unitarily similar to direct sums. This means that there exists an unitary matrix U and matrices $H_1, H_2, K_1, K_2 \in M_3(\mathbb{C})$ such that

$$UHU^* = \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} \text{ and } UKU^* = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}.$$

Connection with real algebraic geometry

Note that F defines a real cubic curve in $\mathbb{CP}^2$ and that $zId - xH - yK$ is a definite determinantal representation of F.

In the terminology of linear matrix inequalities:

- F is a real zero polynomial;
- point $(0,0)$ lies inside the convex set of points $\{(x,y) \in \mathbb{R}^2 \ : \ Id - xH - yK \geq 0\}$ called spectrahedron;
- spectrahedron is bounded by the compact part of the curve.

More constructions of definite determinantal representations of polynomials are due to Netzer, Thom and Quarez (2012 $\rightarrow$).

Every smooth real cubic can be brought into a Weierstrass form by a real change of coordinates

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto P \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \text{ for some } P \in \mathrm{GL}_3(\mathbb{R}).$$

In the new coordinates we get

$$\det(xA_x + yA_y + zA_z) = \left(-yz^2 + x^3 + \alpha xy^2 + \beta y^3\right)^2, \text{ where } \alpha, \beta \in \mathbb{R}.$$

- A_x, A_y, A_z are real linear combinations of H, K, Id and therefore Hermitian;
- $z\,Id - x\,H - y\,K$ is definite, so $A = x\,A_x + y\,A_y + z\,A_z$ is also definite.

The cokernel of A is a rank 2 bundle

When the cokernel is decomposable $\mathcal{L}_1 \oplus \mathcal{L}_2$, then A is equivalent to a block matrix $\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$, where $\mathcal{L}_i$ is the cokernel of A_i.

In other words,

$$\det A_i = -yz^2 + x^3 + \alpha xy^2 + \beta y^3,$$

and both A_i are determinantal representations of order 1.

Vinnikov, 1986–1989

Each 3×3 determinantal representation A_i is unitarily equivalent to one of the two self-adjoint forms

$$\pm\left(x\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}+z\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}+y\begin{bmatrix}\alpha+\frac{3}{4}t_i^2 & is_i & \frac{t_i}{2}\\ -is_i & -t_i & 0\\ \frac{t_i}{2} & 0 & -1\end{bmatrix}\right),$$

where $(s_i, t_i) \in \mathbb{R}^2$ satisfy $-s_i^2 = t_i^3 + \alpha t_i + \beta$.

Moreover, definite self-adjoint determinantal representations of C are exactly those corresponding to the points (s, t) in the compact part of $C(\mathbb{R})$.

Indecomposable vector bundles

Atiyah (1957) classified indecomposable rank r vector bundles on elliptic curves that correspond to r-torsion points.

Rank 2 vector bundles are the cokernels of

$$x\begin{bmatrix} 0&0&0&0&0&1\\ &0&0&0&1&0\\ &&0&1&0&0\\ &&&0&0&0\\ &&&&0&0\\ &&&&&0 \end{bmatrix} + z\begin{bmatrix} 0&0&0&0&1&0\\ &0&0&1&0&0\\ &&0&0&0&0\\ &&&0&0&0\\ &&&&0&0\\ &&&&&0 \end{bmatrix} + y\begin{bmatrix} 0&1&0&\frac{3t^2-2t(1+\lambda)-(1-\lambda)^2}{4}&0&\frac{t-1-\lambda}{2}\\ &0&0&0&-t&0\\ &&0&\frac{t-1-\lambda}{2}&0&-1\\ &&&0&0&0\\ &&&&0&0\\ &&&&&0 \end{bmatrix},$$

for t satisfying $0 = t(t-1)(t-\lambda)$.

Indecomposable vector bundles

Remark: i times the above skew-symmetric representation is self-adjoint.

It is easy to check that these three determinantal representations are **not** definite and can thus not provide a counterexample to Kippenhahn's conjecture.

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