Indecomposable Matrices Defining Plane Cubics

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Outline I

1. Kippenhahn’s conjecture
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   - Determinantal representations of cubics

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   - Proof of Kippenhahn’s conjecture for cubics
Kippenhahn’s conjecture on Hermitian pencils (1951)

Let \( H, K \) be \( n \times n \) complex Hermitian matrices and \( F \in \mathbb{C}[x, y, z] \) a homogeneous polynomial such that

\[
\det(x \ H + y \ K - z \ \text{Id}) = F(x, y, z).
\]

When \( F(x, y, z) \) has a repeated factor, then \( H \) and \( K \) are simultaneously unitarily similar to direct sums: there exists an unitary matrix \( U \) and matrices \( H_i, K_i \in M_{n_i}(\mathbb{C}) \) for some \( 1 \leq n_i < n \) such that

\[
UHU^* = \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} \quad \text{and} \quad UKU^* = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}.
\]

By Burnside’s theorem on matrix algebras, this is equivalent to \( H \) and \( K \) not generating the whole \( M_n(\mathbb{C}) \).
Shapiro (1982) showed that the conjecture holds for $n \leq 5$ and for $n = 6$ in the case that the minimal polynomial is cubic.

Waterhouse (1984) presented a pair of $6 \times 6$ matrices that generate $M_6(\mathbb{C})$ such that $\det(xH + yK - zd)$ has repeated linear factors, thus disproving the general form of Kippenhahn’s conjecture for $n = 6$.

Li and Spitkovsky (1998) constructed another class of counterexamples for $n = 6$.

Laffey (1983) constructed a counterexample for $n = 8$ with quartic minimal polynomial.
The conjecture of Kippenhahn is true for \( n = 6 \) with cubic minimal polynomial.

Let \( H, K \) be \( 6 \times 6 \) complex Hermitian matrices and \( F \) a homogeneous polynomial defining a smooth cubic, such that

\[
\det(xH + yK - zI) = F(x, y, z)^2.
\]

Then \( H \) and \( K \) are simultaneously unitarily similar to direct sums. This means that there exists an unitary matrix \( U \) and matrices \( H_1, H_2, K_1, K_2 \in M_3(\mathbb{C}) \) such that

\[
UHU^* = \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} \quad \text{and} \quad UKU^* = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}.
\]
Let $C$ be an irreducible curve in $\mathbb{CP}^2$ defined by a polynomial $F(x, y, z)$ of degree 3. Every smooth cubic can be brought by a change of coordinates

\[
\begin{bmatrix}
  x \\
  y \\
  z
\end{bmatrix} \mapsto P
\begin{bmatrix}
  x \\
  y \\
  z
\end{bmatrix}, \quad \text{for some } P \in \text{GL}_3(\mathbb{C})
\]

into a Weierstrass form

\[
F(x, y, z) = yz^2 - x(x - y)(x - \lambda y) = 0,
\]

or equivalently

\[
F(x, y, z) = -yz^2 + x^3 + \alpha xy^2 + \beta y^3 = 0,
\]

for some $\lambda \neq 0, 1$ and $\alpha, \beta \in \mathbb{C}$. 
Consider the following question. For given $C$ find a linear matrix

$$A(x, y, z) = x A_x + y A_y + z A_z$$

such that

$$\det A(x, y, z) = c F(x, y, z)^r,$$

where $A_x, A_y, A_z \in M_{3r}$ and $0 \neq c \in \mathbb{C}$. Here $M_{3r}$ is the algebra of all $3r \times 3r$ matrices over $\mathbb{C}$.

We call $A$ a determinantal representation of $C$ of order $r$.

Determinantal representation $A$ is definite if $A(x_0, y_0, z_0)$ is definite at some point $(x_0, y_0, z_0)$. 
Equivalent determinantal representations

Two determinantal representations $A$ and $A'$ are equivalent if there exist $X, Y \in \text{GL}_3(\mathbb{C})$ such that

$$A' = XAY.$$ 

We study:

- self-adjoint representations $A = A^*$ modulo unitary equivalence $Y = X^*$,
- skew-symmetric representations $A = -A^t$ under $Y = X^t$ equivalence.

Obviously, equivalent determinantal representations define the same curve.
Theorem

Consider a linear matrix $A = xA_x + yA_y + zA_z$ with $\det A = F(x, y, z)^r$. When $F$ defines a smooth curve $C$, the cokernel of $A$ is a vector bundle of rank $r$ on $C$.

The conjecture of Kippenhahn is true for $n = 6$ with cubic minimal polynomial.

Let $H, K$ be $6 \times 6$ complex Hermitian matrices and $F$ a homogeneous polynomial defining a smooth cubic, such that

$$\det(xH + yK - zI) = F(x, y, z)^2.$$ 

Then $H$ and $K$ are simultaneously unitarily similar to direct sums. This means that there exists an unitary matrix $U$ and matrices $H_1, H_2, K_1, K_2 \in M_3(\mathbb{C})$ such that

$$UHU^* = \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} \quad \text{and} \quad UKU^* = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}.$$
Connection with real algebraic geometry

Note that $F$ defines a **real cubic curve** in $\mathbb{CP}^2$ and that 
$zId - xH - yK$ is a **definite determinantal representation** of $F$.

In the terminology of linear matrix inequalities:
- $F$ is a **real zero polynomial**;
- point $(0, 0)$ lies inside the convex set of points 
  $\{(x, y) \in \mathbb{R}^2 : Id - xH - yK \geq 0\}$ called **spectrahedron**;
- spectrahedron is bounded by the compact part of the curve.

More constructions of definite determinantal representations of polynomials are due to Netzer, Thom and Quarez (2012).
Every smooth real cubic can be brought into a Weierstrass form by a real change of coordinates

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto P \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \text{ for some } P \in \text{GL}_3(\mathbb{R}).$$

In the new coordinates we get

$$\det(xA_x+yA_y+zA_z) = \left(-yz^2 + x^3 + \alpha xy^2 + \beta y^3\right)^2,$$

where $\alpha, \beta \in \mathbb{R}$.

- $A_x, A_y, A_z$ are real linear combinations of $H, K, \text{Id}$ and therefore Hermitian;
- $z \text{Id} - x H - y K$ is definite, so $A = x A_x + y A_y + z A_z$ is also definite.
The cokernel of $A$ is a rank 2 bundle

When the cokernel is decomposable $\mathcal{L}_1 \oplus \mathcal{L}_2$, then $A$ is equivalent to a block matrix

$$
\begin{bmatrix}
A_1 & 0 \\
0 & A_2
\end{bmatrix},
$$

where $\mathcal{L}_i$ is the cokernel of $A_i$.

In other words,

$$
\det A_i = -yz^2 + x^3 + \alpha xy^2 + \beta y^3,
$$

and both $A_i$ are determinantal representations of order 1.
Each $3 \times 3$ determinantal representation $A_i$ is unitarily equivalent to one of the two self-adjoint forms

$$\pm \left( x \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} + z \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + y \begin{bmatrix} \alpha + \frac{3}{4} t_i^2 & i s_i & \frac{t_i}{2} \\ -i s_i & -t_i & 0 \\ \frac{t_i}{2} & 0 & -1 \end{bmatrix} \right),$$

where $(s_i, t_i) \in \mathbb{R}^2$ satisfy $-s_i^2 = t_i^3 + \alpha t_i + \beta$.

Moreover, definite self-adjoint determinantal representations of $C$ are exactly those corresponding to the points $(s, t)$ in the compact part of $C(\mathbb{R})$. 
Atiyah (1957) classified indecomposable rank $r$ vector bundles on elliptic curves that correspond to $r$-torsion points.

Rank 2 vector bundles are the cokernels of

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + z \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + y \begin{bmatrix} 0 & 1 & 0 & \frac{3t^2 - 2t(1+\lambda) - (1-\lambda)^2}{4} & 0 \\ 0 & 0 & 0 & 0 & -t \\ 0 & 0 & 0 & \frac{t-1-\lambda}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

for $t$ satisfying $0 = t(t - 1)(t - \lambda)$.
Remark: $i$ times the above skew-symmetric representation is self-adjoint.

It is easy to check that these three determinantal representations are not definite and can thus not provide a counterexample to Kippenhahn’s conjecture.
Kippenhahn’s conjecture
Determinantal representations
Vector bundles

Cokernels are vector bundles of rank \( r \)
Proof of Kippenhahn’s conjecture for cubics


