Length realizability problem for pairs of quasi-commuting matrices

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Following [1, 2], we define *the length of a finite system of generators* $\mathcal{S}$ of a given finite-dimensional algebra $\mathcal{A}$ over a field as the smallest number k such that products in $\mathcal{S}$ of length not greater than k generate $\mathcal{A}$ as a vector space. For the generating sets of the full matrix algebra $M_n(\mathbb{F})$ the problem of computing the length as a function of n is studied since 1984 and is still an open problem. However, there exist some good bounds for the lengths of matrix sets satisfying some additional conditions. In this talk we discuss the length evaluation problem for quasi-commuting pairs of matrices (we say that A, B in $M_n(\mathbb{F})$ *quasi-commute* if AB and BA are linearly dependent).

First we single out two special classes: (I) commuting pairs and (II) quasi-commutative, non-commuting pairs with a nilpotent product. We show that in each of these cases $l(\mathcal{S}) \leq n-1$ and, moreover, for any $l = 1, \dots, n-1$, each of these two classes contains a pair of matrices with length l.

If a quasi-commuting pair $\mathcal{S} = \{A, B\} \subset M_n(\mathbb{F})$ does not belong to the class (I)∪(II), then $AB = \varepsilon BA$ where the commutativity factor ε is a primitive k-th root of unity for some $k \leq n$. We will show that in this case the situation is very different from the commutative and nilpotent case. We provide sharp upper and lower bounds for the length of such pairs depending on n, k and the algebraic multiplicity of 0 as an eigenvalue of AB. We show how the interval between these extremal values is divided into intervals of realizable values for the length and “gaps”, i.e. non-realizable values.

This is a joint work with ALEXANDER GUTERMAN (Lomonosov Moscow State University, Russia) and VOLKER MEHRMANN (Technische Universität Berlin, Germany). The talk is partially based on our papers [3, 4].

REFERENCES

[1] A. Paz, *An application of the Cayley–Hamilton theorem to matrix polynomials in several variables*, Linear Multilinear Algebra, **15**, 1984, 161–170.
[2] C.J. Pappacena, *An upper bound for the length of a finite-dimensional algebra*, J. Algebra, **197**, 1997, 535–545.
[3] A.E. Guterman, O.V. Markova, *The realizability problem for values of the length function for quasi-commuting matrix pairs*, Zap. Nauchn. Sem. POMI. **439**, 2015, 59–73.
[4] A.E. Guterman, O.V. Markova, V. Mehrmann, *Lengths of quasi-commutative pairs of matrices*, Linear Algebra Appl. **498**, 2016, 450–470.