Bounds on tensor norms via tensor partitions

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The spectral norm and the nuclear norm of a matrix are evidently important in many branches of mathematics as well as in various practical applications. They are easy to compute from the singular value decompositions. In recent years, due to the surge of research on studying various tensor problems and multilinear algebra, the use of tensor spectral norm and tensor nuclear norm are widely seen, in particular in tensor decompositions and tensor completions. However, these tensor norms are NP-hard to compute in general. In this work, we study tensor norms from a new perspective. We introduce several concepts of tensor partitions, generalizing the concept of block tensor in the literature. Neat bounds on the spectral norm and the nuclear norm of a tensor based on arbitrary partitions are established. Specifically, given any tensor $\mathcal{T}$ that is partitioned into a set of subtensors $\{\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_m\}$, its spectral norm $\|\mathcal{T}\|_\sigma$ and its nuclear norm $\|\mathcal{T}\|_*$ can be bounded as follows:

$$\|(\|\mathcal{T}_1\|_\sigma, \|\mathcal{T}_2\|_\sigma, \ldots, \|\mathcal{T}_m\|_\sigma)\|_\infty \le \|\mathcal{T}\|_\sigma \le \|(\|\mathcal{T}_1\|_\sigma, \|\mathcal{T}_2\|_\sigma, \ldots, \|\mathcal{T}_m\|_\sigma)\|_2$$
$$\|(\|\mathcal{T}_1\|_*, \|\mathcal{T}_2\|_*, \ldots, \|\mathcal{T}_m\|_*)\|_2 \le \|\mathcal{T}\|_* \le \|(\|\mathcal{T}_1\|_*, \|\mathcal{T}_2\|_*, \ldots, \|\mathcal{T}_m\|_*)\|_1,$$

where $\|\cdot\|_p$ stands for the L_p-norm of a vector for $1 \le p \le \infty$. These intuitive bounds are tight in general and can be extended to the tensor spectral p-norm and nuclear p-norm for any $1 \le p \le \infty$. We also study the relation of the norm of a tensor, the norms of matrix unfoldings of the tensor, and bounds via the norms of matrix slices of the tensor. Various bounds of the tensor norms in the literature are implied by our results.