**Local-to-global properties for matrix semigroups and quasidiagonal operators and other stuff**

LAURENT W. MARCOUX, HEYDAR RADJAVI
*Department of Pure Mathematics*
*University of Waterloo (Canada)*
LWMarcoux@uwaterloo.ca, HRadjavi@uwaterloo.ca

1. If we know a local property for a multiplicative semigroup of matrices, what can we say about the semigroup? For example, let $\mathcal{S}$ be a semigroup of $n \times n$ complex matrices and let $P$ be an orthogonal projection matrix. We may know something about the collection

$$P\mathcal{S}P = \{PAP : A \in \mathcal{S}\},$$

say, it is commutative. Does this imply something about the structure of $\mathcal{S}$? If $P\mathcal{S}P$ is finite or bounded, it is already known that so is $\mathcal{S}$ itself. If we assume commutativity for $P\mathcal{S}P$ (in which case we assume, of course, that the rank of $P$ is more than one) we can show that $\mathcal{S}$ is reducible, i.e., it has a common invariant subspace. What else can we say about the structure? There are other variations of the local commutativity question.

The case where $P$ has rank 1 has been studied by several authors in recent years, and some of the talks given during the conference will address this, and may generate new questions.

2. There are known results of the general form that certain approximate equalities yield the corresponding exact equality. For example, a result of Bernik and Radjavi (2005) says that if in group of unitary $n \times n$ matrices if

$$\|AB - BA\| < \sqrt{3}$$

for all $A$ and $B$ in the group, then the group is commutative. Or, a result of Marcoux, Mastnak and Radjavi (2007) says, among other things, that if $A$ and $B$ are two invertible $n \times n$ matrices such that

$$\operatorname{tr}(A^k - B^k) < 1$$

for all integers k, then $A$ and $B$ have the same spectra (with the same multiplicities). Is it possible to weaken the assumption by restricting $k$ to a bounded subset of integers in this statement? The short answer is no, because you can take $A$ and $B$ close to the identity. But we are looking for long answers: what if you assume something about the norms of $A$ and $B$?

3. Let $M$ be a nonnegative matrix (i.e., all its entries are nonnegative). If the diagonal of $M$ consists exactly of its eigenvalues with the right multiplicities, then $M$ is triangular after a similarity by a permutation. This was extended to infinite-dimensional setting by Bernik, Marcoux and Radjavi (2012). What about general operators?not necessarily nonnegative? Again we are looking for long answers!

4. A result in Livshits, MacDonald and Radjavi (2011) is the following: let $\mathcal{S}$ be a semigroup of nonnegative matrices (in the sense of Section 3 above) and

assume $\mathcal{S}$ is indecomposable (that is, it has no simultaneous nontrivial invariant subspace spanned by the standard basis vectors). If the diagonals of all members of $\mathcal{S}$ belong to $\{0,1\}$, then after a simultaneous diagonal similarity all the entries of all members of $\mathcal{S}$ are in $\{0,1\}$. What happens if we replace the set $\{0,1\}$ in the hypotheses by another set with a structure?

5. We haven't given the next problem much thought at all, and it might be very easy. (This is not to say that we have thought about all of the above problems that deeply...)

   Suppose that $\mathcal{T} = \mathcal{T}_n(\mathbb{C}) \subseteq \mathbb{M}_n(\mathbb{C})$ is the algebra of upper triangular matrices. The annihilator of $\mathcal{T}$, namely $\mathcal{T}^\perp := \{X \in \mathbb{M}_n(\mathbb{C}) : \mathrm{tr}(T^*X) = 0\}$ is then the set of all *strictly* lower-triangular matrices. Observe that this means that $\mathcal{T}^\perp$ is itself an *algebra*.

   (a) For which algebras $\mathcal{A} \subseteq \mathbb{M}_n(\mathbb{C})$ is it also the case that $\mathcal{A}^\perp$ is again an algebra? In particular, must $\mathcal{A}$ be a finite-dimensional "nest algebra" (i.e. the full set of block-upper triangular matrices with respect to some basis of the Hilbert space)?

   (b) Is there an intrinsic characterization of those subspaces $\mathcal{L} \subseteq \mathbb{M}_n(\mathbb{C})$ which are the annihilators of *some* $\mathrm{Alg}(T)$, the algebra generated by a fixed $T \in \mathbb{M}_n(\mathbb{C})$? For example, a necessary condition is that such a space $\mathcal{L}$ must have dimension at least equal to $n^2 - n$, as $\dim \mathrm{Alg}(T) \le n$ for all $T \in \mathbb{M}_n(\mathbb{C})$.

6. SPECHT'S THEOREM IN A $C^*$-ALGEBRA? Suppose that $\mathcal{A}$ is a simple $C^*$-algebra with a unique tracial state $\tau$ and that $a, b \in \mathcal{A}$ satisfy

$$\tau(p(a, a^*)) = \tau(p(b, b^*))$$

   for all polynomials $p(x, y)$ in two non-commuting variables $x$ and $y$. Is *a approximately unitarily equivalent* to $b$? That is, does there exist a sequence $(u_n)_n$ of unitary elements of $\mathcal{A}$ so that $b = \lim_n u_n^* a u_n$?

   A test case for this problem would be the case where $\mathcal{A}$ is a *uniformly hyperfinite* (i.e. a (UHF)) $C^*$-algebra.

**Three problems on quasidiagonality**

Here are three problems that I (Laurent) have thought about on and off over the years. I am offering these problems up to a wider audience (you or your students) in the hope that I will learn the answer to these questions before I retire (which I expect to do before my esteemed colleague, Heydar Radjavi).

Let $\mathcal{H}$ be an infinite-dimensional, separable, complex Hilbert space. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be **block-diagonal** (we write $T \in (BD)$) if there exists a bounded sequence $(T_n)_n$ of matrices [each $T_n \in \mathbb{M}_{k_n}(\mathbb{C})$ for some $k_n \ge 1$] such that $T$ is unitarily equivalent to $\oplus_{n=1}^\infty T_n$.

An operator $Q \in \mathcal{B}(\mathcal{H})$ is said to be **quasidiagonal** (we write $T \in (QD)$) if it satisfies any one of the three following (equivalent) conditions:

(a) $Q \in \overline{(BD)}$; i.e. $Q$ is a limit of block-diagonal operators;
(b) $Q = B + K$ for some $B \in (BD)$ and $K \in \mathcal{B}(\mathcal{H})$ a compact operator;
(c) Given $\varepsilon > 0$ there exist $B_\varepsilon \in (BD)$ and $K_\varepsilon$ a compact operator with $\|K_\varepsilon\| < \varepsilon$ so that $T = B_\varepsilon + K_\varepsilon$.

A. Suppose that $Q \in (QD)$ and that $Q$ is **quasinilpotent** - i.e. the spectrum $\sigma(Q) = \{0\}$. Is $Q$ the limit of block-diagonal nilpotent operators? It is known that it suffices to consider the case where $Q$ is itself block-diagonal (and quasinilpotent).

It is important to note that the approximating nilpotent block-diagonal operators need not be block-diagonal with respect to the same decomposition of the Hilbert space that block-diagonalizes $Q$.

KNOWN FACTS:

- $Q \oplus T$ is a limit of block-diagonal nilpotent operators in $\mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ whenever $T \in \mathcal{B}(\mathcal{H})$ is a limit of block-diagonal nilpotent operators. In particular, $Q \oplus 0$ is a limit of block-diagonal nilpotent operators.
- If $N \in \mathcal{B}(\mathcal{H})$ is a normal operator, then $N$ is a limit of block-diagonal nilpotent operators if and only if $\sigma(N)$ is connected and contains 0.

One approach to this problem is to try to solve the following matrix problem: let $T \in \mathcal{B}(\mathbb{C}^n)$ be a norm-one operator and $k \geq 1$. If $\varepsilon := \|T^k\|^{1/k}$, find the distance from $T$ to the set of nilpotent matrices in $\mathcal{B}(\mathbb{C}^n)$ in terms of $\varepsilon$ and $n$. In fact, thanks to a result of T. Loring on the lifting of nilpotent elements in quotients of $C^*$-algebras, the estimate can be made independent of $n$.

B. Let $(DSN) = \{\oplus_n M_n \in (BD) : \text{ each } M_n \text{ is nilpotent}\}$, so that $D \in (DSN)$ precisely if $D$ is (unitarily equivalent to) a direct sum of (a bounded sequence of) nilpotent matrices. Note that the order of nilpotence of $M_n$ depends upon $n$, so that $D$ need not be nilpotent itself - e.g. $D = \oplus_n J_n$ where $J_n$ is the $n \times n$ Jordan cell has spectrum equal to the closed unit disk centred at the origin in $\mathbb{C}$.

Let $(ZTR) = \{\oplus_n Z_n \in (BD) : \operatorname{tr}(Z_n) = 0 \text{ for all } n \geq 1\}$, so that $Z \in (ZTR)$ if and only if $Z$ is (unitarily equivalent to) a direct sum of (a bounded sequence of) matrices, each of whose trace is zero.

Let $(BDN) = \{B \in (BD) : \text{there exists } k \geq 1 \text{ so that } B^k = 0\}$. Thus $B \simeq \oplus_n B_n$, and there exists $k \geq 1$ so that $B_n^k = 0$ for all $n \geq 1$.

It is routine to verify that

$$(BDN) \subseteq (DSN) \subseteq (ZTR).$$

Question: is $(ZTR) \subseteq \overline{(BDN)}$? (The *real* question is to characterize $\overline{(BDN)}$.)

C. It is routine to verify the following:
(a) If $A, B \in (QD)$, then $A \oplus B \in (QD)$. More generally, if $X_n \in (QD)$ for all $n \geq 1$ and $\sup_n \|X_n\| < \infty$, then $\oplus_n X_n \in (QD)$.
(b) If $T \in (QD)$ and $K \in \mathcal{B}(\mathcal{H})$ is a compact operator, then $T + K \in (QD)$.

It is also true (but not as routine) that

(c) If $N \in \mathcal{B}(\mathcal{H})$ is a normal operator, then (by the Weyl-von Neumann-Berg Theorem), $N = D + K$ for some diagonalizable operator $D$ and some compact operator $K$. By (b) above and the fact that a diagonal operator is trivially block-diagonal, $N \in (QD)$.

(d) Suppose that $E \in \mathcal{B}(\mathcal{H})$ is **essentially normal** (i.e. $E^*E - EE^*$ is a compact operator), and that $\|E\| \leq 1$. If $N \in \mathcal{B}(\mathcal{H})$ is a normal operator with $\sigma(N) = \{z \in \mathbb{C} : |z| \leq 1\}$, then (by the Brown-Douglas-Fillmore Theorem), $E \oplus N = M + K$ for some normal operator $M$ and some compact operator $K$, so that $E \oplus N \in (QD)$.

Suppose that $T = Q \oplus E$ has norm equal to 1, and that $Q$ is quasidiagonal and $E$ is essentially normal. Suppose that $N \in \mathcal{B}(\mathcal{H})$ is a normal operator with $\sigma(N) = \{z \in \mathbb{C} : |z| \leq 1\}$.

From (d) above, $T \oplus N = Q \oplus (E \oplus N) = Q \oplus (M + K)$ is a direct sum of two quasidiagonal operators, and hence $T$ is quasidiagonal.

Is the converse true? That is, suppose that $T \in \mathcal{B}(\mathcal{H})$, $\|T\| \leq 1$, and $T \oplus N$ is quasidiagonal, where $N$ is the normal operator above. Must $T$ be a (compact perturbation) of an operator of the form $Q \oplus E$, where $Q$ is quasidiagonal and $E$ is essentially normal?

The answer is known to be "yes" if $T$ is a weighted shift operator.