Positive maps

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A linear map $\Phi\colon M_n(\mathbb{R}) \to M_m(\mathbb{R})$ is called $*$-*linear* if $\Phi(X^T) = \Phi(X)^T$ for each $X \in M_n(\mathbb{R})$. A $*$-linear map $\Phi\colon M_n(\mathbb{R}) \to M_m(\mathbb{R})$ is called *positive* if it maps positive semidefinite matrices to positive semidefinite matrices. Note that a $*$-linear map on the full matrix space is positive if and only if its restriction to the subspace of symmetric matrices, $\Phi\colon \mathrm{Sym}_n \to \mathrm{Sym}_m$, is positive. For any positive integer k, a $*$-linear map $\Phi\colon M_n(\mathbb{R}) \to M_m(\mathbb{R})$ induces a $*$-linear map $\Phi_k\colon M_{kn} \to M_{km}$ defined by

$$\Phi_k([X_{ij}]_{i,j=1}^k) = [\Phi(X_{ij})]_{i,j=1}^k.$$

The linear map Φ is called *k-positive* if Φ_k is positive. Φ is called *completely positive* if it is k-positive for each positive integer k.

The space of linear maps $\Phi\colon \mathrm{Sym}_n \to \mathrm{Sym}_m$ is isomorphic to the space of biquadratic forms in $n+m$ variables via the isomorphism

$$\Phi \mapsto p_\Phi(\mathtt{x},\mathtt{y}) = \mathtt{y}^T\Phi(\mathtt{x}\mathtt{x}^T)\mathtt{y}.$$

Additionally, Φ is positive if and only the polynomial p_Φ is nonnegative on $\mathbb{R}^n \times \mathbb{R}^m$, and Φ is completely positive if and only if p_Φ is a sum of squares of bilinear forms. Investigating the difference between the convex cones of positive and completely positive maps is therefore the same as investigating the difference between the convex cones of nonnegative and SOS biquadratic forms. The two cones are known to be equal only if $m = 2$ or $n = 2$. However, only few examples of positive maps that are not completely positive are known, see e.g., [2, 3, 5, 6, 7]. There is also an algorithm for constructing such maps in [4], but the positive maps obtained from that algorithm are generically not extremal, i.e., they can be written as a sum of two positive maps.

The aim of the working group is to consider some of the following problems:

- Construct new types of extremal positive maps. In particular, in dimension 3 the above mentioned examples are all of the form

$$\begin{bmatrix} x & y & z \\ y & w & v \\ z & v & u \end{bmatrix} \mapsto \begin{bmatrix} a_1x + a_2w + a_3u & a_4y & a_5z \\ a_4y & a_6x + a_7w + a_8u & a_9v \\ a_5z & a_9v & a_{10}x + a_{11}w + a_{12}u \end{bmatrix}.$$

  We would like to know whether in dimension 3 all extremal positive maps that are not completely positive are of that form.
- Quarez [6] initiated the study of the number and configuration of real zeros of nonnegative biquadratic forms with finitely many real zeros. In the case $m = n = 3$ the maximum number of real zeros is 10 [1], but the possible configurations are not known. Moreover, the examples in [1] are the only known

nonnegative biquadratic forms with 10 real zeros. Can we construct more examples? Also, there are no examples of nonnegative biquadratic forms with many real zeros if $m > 3$ or $n > 3$.

- If $\Phi\colon \mathrm{Sym}_n \to \mathrm{Sym}_m$ is a positive map, then $\det \Phi(\mathtt{x}\mathtt{x}^{\mathrm{T}}) \geq 0$ for each $\mathtt{x} \in \mathbb{R}^{\mathtt{n}}$, i.e., $\det \Phi(\mathtt{x}\mathtt{x}^{\mathrm{T}})$ is a nonnegative polynomial. We would like to know if all nonnegative polynomials are of this form, and if not, what are the obstructions. In particular, the polynomials $\det\left(\frac{1}{(t^2-1)^{\frac{2}{3}}}\Phi_t(\mathtt{x}\mathtt{x}^{\mathrm{T}})\right)$ constructed in [1] converge to the Robinson's polynomial
$$x_1^6 + x_2^6 + x_3^6 - x_1^4x_2^2 - x_1^4x_3^2 - x_2^4x_1^2 - x_2^4x_3^2 - x_3^4x_1^2 - x_3^4x_2^2 + 3x_1^2x_2^2x_3^2,$$
but the maps $\left(\frac{1}{(t^2-1)^{\frac{2}{3}}}\Phi_t\right)$ do not converge. Can the Robinson's polynomial still be written as $\det \Phi(\mathtt{x}\mathtt{x}^{\mathrm{T}})$ for some positive map Φ? What about the Motzkin's polynomial
$$x_1^4x_2^2 + x_1^2x_2^4 + x_3^6 - 3x_1^2x_2^2x_3^2?$$

References

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