Hans Schneider ILAS Lecture

Circles in the spectrum and numerical ranges

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We prove that a bounded linear Hilbert space operator has the unit circle in its essential approximate point spectrum if and only if it admits an orbit satisfying certain orthogonality and almost-orthogonality relations.

As consequences, we derive in particular wide generalizations of Arveson's theorem as well as show that the weak convergence of operator powers implies the uniform convergence of their compressions on an infinite-dimensional subspace.

This is a joint work with YURI TOMILOV (Nicolaus Copernicus University and Polish Academy of Sciences).

Invited talks

Wiener's lemma along primes and other subsequences

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Inspired by ergodic theorems along subsequences, we study the validity of Wiener's lemma and extremal behavior of measures μ on the unit circle concerning their Fourier coefficients $\hat{\mu}(k_n)$ along subsequences (k_n) of $\mathbb{N}$, with focus on arithmetic subsequences such as polynomials, primes and polynomials of primes. We also discuss consequences for orbits of operators extending results of J. Goldstein and B. Nagy. As an application of the general results we shall prove among others the following facts which may seem surprising. Denote by p_n the n^{th} prime:

(1) If T is a (linear) contraction on a Hilbert space and $x \in H \setminus \{0\}$ is such that $|(T^{p_n}x|x)| \to \|x\|^2$ as $n \to \infty$, then x is an eigenvector of T to a unimodular eigenvalue.
(2) If T is a power bounded operator on a Banach space E and $x \in E \setminus \{0\}$ is such that $|\langle T^{p_n}x, x'\rangle| \to |\langle x, x'\rangle|$ as $n \to \infty$ for every $x' \in E'$, then x is an eigenvector of T to a unimodular eigenvalue.
(3) If T is a power bounded operator on a Banach space E with $T^{p_n} \to I$ in the weak operator topology, then $T = I$.

This is a joint work with TANJA EISNER (University of Leipzig).

Krauter conjecture on permanents is true

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Permanent function is very useful in algebra and combinatorics. The central role in these investigations is played by $(0,1)$ or $(-1,1)$ matrices, which are important for the combinatorial applications of permanent. The following problem related to the permanent of $(-1,1)$ matrices was posed in 1974.

Problem 1. (Wang, [2, Problem 2].) *Let $A \in M_n(\pm 1)$ be a nonsingular matrix. Is there a decent upper bound for* $|\mathrm{per}\,(A)|$*?*

Kräuter in the paper [1] formulated the following conjecture which provides a possible upper bound for the values of the permanent function for matrices from $M_n(\pm 1)$ via the rank function.

Conjecture 1. (Kräuter, [1, Conjecture 5.2], 1985.) *Let* $n \geq 5$, $A \in M_n(\pm 1)$ *and* $\operatorname{rk}(A) = r+1$ *for some* r, $0 \leq r \leq n-1$. *Then* $|\operatorname{per}(A)| \leq \operatorname{per}(D(n,r))$, *where* $D(n,r) = (d_{ij}) \in M_n(\pm 1)$ *is defined by* $d_{ij} = -1$ *if* $i = j$ *and* $j \in \{1, \ldots, r\}$, $d_{ij} = 1$ *otherwise. The equality holds iff the matrix* A *can be obtained from* $D(n,r)$ *by the transposition, row or column permutations and multiplications of rows or columns by* -1.

We show that this conjecture is true.

This is a joint work with Mikhail Budrevich (Lomonosov Moscow State University).

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On projections arising from isometries with finite spectrum on Banach spaces

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If P is an orthogonal projection on a Hilbert space, then it can be written in the form $P = \frac{I+T}{2}$ for an isometry (a unitary operator) T satisfying $T^2 = I$. When looking for a suitable generalization of orthogonal projections in the Banach space setting, the main task is to get rid of the involution in defining an orthogonal projection. One way is to consider Banach space projections that can be written as the average of the identity with an isometric reflection. If T is an isometric reflection then $\sigma(T) = \{1, -1\}$, and for $P = \frac{I+T}{2}$ we have $T = P - (I-P)$. More generally, if T is an isometry such that $\sigma(T) = \{1, \lambda\}$ with $\lambda \neq 1$, then there exists a projection P such that $T = P + \lambda(I-P)$; in this case P is called a generalized bicircular projection. One can also consider generalized n-circular projections that arise from isometries with n distinct eigenvalues. In this talk we shall describe the structure of generalized n-circular projections on some important complex Banach spaces, mostly in the case $n = 2$, but also a few for $n \geq 3$.

On the lengths of some generating sets of matrix algebras

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Suppose that $\mathcal{S}$ is a set of generators of the full algebra $\mathcal{A}$ of $n \times n$ complex matrices. The least integer k for which the monomials in the elements of $\mathcal{S}$ of degree at most k span $\mathcal{A}$ is called the *length* of $\mathcal{S}$ .A conjecture of Paz (LAMA **15** (1984) 161-170) states that the length of $\mathcal{S}$ is at most $2n-2$. If true, the conjecture is best possible, since examples are known where the bound $2n-2$ is achieved.

In this talk, we will consider a number of generating sets, mostly consisting of matrices with quadratic or cubic minimal polynomials, which one might expect to have large length, and verify Paz's conjecture for them. We will also discuss approaches to proving weaker versions of the conjecture.

This work is based on a collaboration with HELENA ŠMIGOC (University College Dublin) and ALEXANDER GUTERMAN and OLGA MARKOVA (Moscow State University).

Matrix problems in quantum information science

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We will discuss some recent matrix results and problems on matrix inequalities, special matrices, matrix transformations, arising in quantum information science.

Vector states on operator semigroups

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Let $\mathcal{S}$ be a multiplicative semigroup of bounded linear operators on a complex Hilbert space $\mathcal{H}$, and let Ω be the range of a vector state on $\mathcal{S}$ so that $\Omega = \{\langle S\xi, \xi\rangle : S \in \mathcal{S}\}$ for some fixed unit vector $\xi \in \mathcal{H}$. We study the structure of sets of cardinality two coming from irreducible semigroups $\mathcal{S}$. This leads us to sufficient conditions for reducibility and, in some cases, for the existence of common fixed points for $\mathcal{S}$. This is made possible by a thorough investigation of the structure of maximal families $\mathcal{F}$ of unit vectors in $\mathcal{H}$ with the property that there exists a fixed constant $\rho \in \mathbb{C}$ for which $\langle x, y \rangle = \rho$ for all distinct pairs x and y in $\mathcal{F}$.

This is joint work with HEYDAR RADJAVI (University of Waterloo) and BAMDAD YAHAGHI (Golestan Unversity).

Distributing trace

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It is well known that, up to (unitary) similarity, the trace of a matrix can be aribitrarily distributed along its diagonal. In joint work in progress with G. MACDONALD (University of Prince Edward Island), L. MARCOUX (University of Waterloo), M. OMLADIČ (University of Ljubljana), and H. RADJAVI (University of Waterloo) we study the problem for collections of matrices. We obtain, among other things, a complete classification of $*$-subalgebras of n-by-n matrices that are unitarily similar to a subalgebra of the subspace of matrices whose diagonal part is scalar (i.e., all diagional entries are equal).

Simultaneous versions of Perron-Frobenius and Wielandt results

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The classical theorem of Perron and Frobenius shows how much insight can be gained into the structure of an operator if it is assumed positive – or, the entries of its matrix in a suitable basis are all nonnegative. That of Wielandt gives a sufficient condition for "positivization" of certain operators on a complex linear space, i.e., a basis change to turn them to positive operators. These elegant results have attracted many authors and inspired extensions, to infinite dimensions on the one hand, and to simultaneous situations on the other:

(a) What structural information can be obtained for semigroups of positive operators?
(b) Under what conditions can we positivize a semigroup of operators simultaneously?

Some of the most recent simultaneous versions will be discussed.

Sylvester equation in triangular operator algebras

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The talk will discuss the spectrum of the operator $X \mapsto AX + XB$ in triangular operator algebras.

An equivalence result in the symmetric nonnegative inverse eigenvalue problem

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We say that a list of real numbers is "symmetrically realisable" if it is the spectrum of some (entrywise) nonnegative symmetric matrix. The Symmetric Nonnegative Inverse Eigenvalue Problem (SNIEP) is the problem of characterising all symmetrically realisable lists.

We present a recursive method for constructing symmetrically realisable lists. The properties of the realisable family we obtain allow us to make several novel connections between a number of sufficient conditions developed over forty years, starting with the work of Fiedler in 1974. We show that essentially all previously known sufficient conditions are either contained in or equivalent to the family we are introducing.

This is a joint work with RICHARD ELLARD (University College Dublin).

Unbounded convergences in vector and Banach lattices

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Let τ be a mode of convergence of nets in a vector lattice X. We define its "unbounded" counterpart as follows: $x_\alpha \xrightarrow{\mathrm{u}\tau} x$ if $|x_\alpha - x| \wedge u \xrightarrow{\tau} x$ for every $u \geq 0$. In my talk, I will present an overview of recent results by various authors on unbounded order (uo) convergence on vector lattices and unbounded norm (un) convergence on Banach lattices. For sequences in most function spaces, these convergences agree with convergence almost everywhere and with convergence in measure, respectively. Hence, one can think of uo and un convergences as generalizations of convergences everywhere and in measure, respectively. This allows one to extend various facts of measure theory and L_p-spaces to the much broader setting of function spaces and Banach lattices.

While uo convergence is not topological, un convergence is. We will discuss properties of un topology. In particular, un topology is metrizable iff the space has a quasi-interior point. I will also discuss extending un topology to the universal completion of the space.

Contributed talks

Indecomposable matrices defining plane cubics

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Consider a Weierstrass cubic in $\mathbb{CP}^2$ defined by the polynomial

$$F(x,y,z) = yz^2 - x(x-y)(x-\lambda y) = 0,$$

for some $\lambda \neq 0, 1$. For given F we find all linear matrices

$$A(x,y,z) = x\,A_x + y\,A_y + z\,A_z$$

such that

$$\det A(x,y,z) = c\,F(x,y,z)^2,$$

where A_x, A_y, A_z are 6×6 matrices over $\mathbb{C}$ and $0 \neq c \in \mathbb{C}$. In other words, we find all (decomposable and indecomposable) 6×6 linear determinantal representations of Weierstrass cubics.

As a corollary we verify the Kippenhahn conjecture for 6×6 matrices.

Extremal non-convertible fully indecomposable $(0,1)$-matrices

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Permanent is a function which is similar to determinant by its definition but considerably different by its properties. Permanent of $(0,1)$-matrices has an important role as a computing function in combinatorics. In this work we restrict our attention to the permanent function on $(0,1)$-matrices only.

In [1] it was proved that any fully indecomposable not convertible $(0,1)$-matrix A of order n has at least $2n+3$ positive entries. In this talk we present the description of all such matrices with the minimal possible number of non-zero entries in matrix terms and in graph terms.

Structure of fully indecomposable non-convertible $(0,1)$-matrix with $2n+3$ positive elements is similar to sparse circulant matrices. Using this fact we compute permanent of all such matrices and show that these matrices give a series of examples of non-convertible matrices which satisfy the conditions: a matrix can not be represented in upper block triangular form and a matrix has minimal possible permanent.

This is a joint work with GREGOR DOLINAR (University of Ljubljana), ALEXANDER E. GUTERMAN (Lomonosov Moscow State University) and BOJAN KUZMA (University of Ljubljana).

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Some open questions about Kronecker quotients

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This talk will review results on the existence of different types of Kronecker quotients, and consider some open questions regarding uniform Kronecker quotients.

Consider the vector space $\mathcal{M}_{m,n}$ of $m \times n$ matrices over some field $\mathbb{F}$ and the Kronecker product $A \otimes B \in \mathcal{M}_{ms,nt}$ of matrices $A \in \mathcal{M}_{m,n}$ and $B \in \mathcal{M}_{s,t}$.

We may define a quotient operation $\oslash : \mathcal{M}_{ms,nt} \times \mathcal{M}_{s,t} \to \mathcal{M}_{m,n}$. A Kronecker quotient $\oslash$ obeys $(A \otimes B) \oslash B = A$ for all matrices A and $B \neq 0$. In particular, a *uniform Kronecker quotient* $\oslash$ is linear in its left argument and obeys

$$(A \otimes B) \oslash C = (B \oslash C)A$$

when B and C have the same size (so that $B \oslash C \in \mathbb{F}$).

For each uniform Kronecker quotient, there exists $Q : \mathcal{M}_{s,t} \to \mathcal{M}_{s,t}$ such that, for $M \in \mathcal{M}_{ms,nt}$,

$$M \oslash B = \mathrm{tr}_2((I_m \otimes Q(B))^T M)$$

where I_m is the $m \times m$ identity matrix and the partial trace tr_2 is taken over the $t \times t$ matrices in $\mathcal{M}_{m,n} \otimes \mathcal{M}_{t,t}$. This is known as a partial Frobenius product.

Linear spaces of symmetric nilpotent matrices

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In 1958 Gerstenhaber showed that if $\mathcal{L}$ is a subspace of the vector space of the square matrices of order n over some field $\mathbb{F}$, consisting of nilpotent matrices, and the field $\mathbb{F}$ is sufficiently large, then the maximal dimension of $\mathcal{L}$ is $\frac{n(n-1)}{2}$, and if this dimension is attained, then the space $\mathcal{L}$ is triangularizable. Linear spaces of symmetric matrices seem to be first studied by Meshulam in 1989 in view of the bound of their rank. Although it seems unnatural to ask when a linear space of symmetric matrices is made of nilpotents and when it is triangular, we find a way to do so by going to an equivalent notion for symmetric matrices, i.e. persymmetric matrices. We develop a theory that enables us to prove extensions of some beautiful classical triangularizability results to the case of symmetric matrices. Not only the Gerstenhaber's result, but also Engel, Jacobson and Radjavi theorems can be extended. We also study maximal linear spaces of symmetric nilpotents of smaller dimension.

This is a joint work with MATJAŽ OMLADIČ (Institute of mathematics, physics and mechanics, Ljubljana).

Bounds on tensor norms via tensor partitions

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The spectral norm and the nuclear norm of a matrix are evidently important in many branches of mathematics as well as in various practical applications. They are easy to compute from the singular value decompositions. In recent years, due to the surge of research on studying various tensor problems and multilinear algebra, the use of tensor spectral norm and tensor nuclear norm are widely seen, in particular in tensor decompositions and tensor completions. However, these tensor norms are NP-hard to compute in general. In this work, we study tensor norms from a new perspective. We introduce several concepts of tensor partitions, generalizing the concept of block tensor in the literature. Neat bounds on the spectral norm and the nuclear norm of a tensor based on arbitrary partitions are established. Specifically, given any tensor $\mathcal{T}$ that is partitioned into a set of subtensors $\{\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_m\}$, its spectral norm $\|\mathcal{T}\|_\sigma$ and its nuclear norm $\|\mathcal{T}\|_*$ can be bounded as follows:

$$\|(\|\mathcal{T}_1\|_\sigma, \|\mathcal{T}_2\|_\sigma, \dots, \|\mathcal{T}_m\|_\sigma)\|_\infty \le \|\mathcal{T}\|_\sigma \le \|(\|\mathcal{T}_1\|_\sigma, \|\mathcal{T}_2\|_\sigma, \dots, \|\mathcal{T}_m\|_\sigma)\|_2$$
$$\|(\|\mathcal{T}_1\|_*, \|\mathcal{T}_2\|_*, \dots, \|\mathcal{T}_m\|_*)\|_2 \le \|\mathcal{T}\|_* \le \|(\|\mathcal{T}_1\|_*, \|\mathcal{T}_2\|_*, \dots, \|\mathcal{T}_m\|_*)\|_1,$$

where $\| \cdot \|_p$ stands for the L_p-norm of a vector for $1 \leq p \leq \infty$. These intuitive bounds are tight in general and can be extended to the tensor spectral p-norm and nuclear p-norm for any $1 \leq p \leq \infty$. We also study the relation of the norm of a tensor, the norms of matrix unfoldings of the tensor, and bounds via the norms of matrix slices of the tensor. Various bounds of the tensor norms in the literature are implied by our results.

Length realizability problem for pairs of quasi-commuting matrices

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Following [1, 2], we define *the length of a finite system of generators* $\mathcal{S}$ of a given finite-dimensional algebra $\mathcal{A}$ over a field as the smallest number k such that products in $\mathcal{S}$ of length not greater than k generate $\mathcal{A}$ as a vector space. For the generating sets of the full matrix algebra $M_n(\mathbb{F})$ the problem of computing the length as a function of n is studied since 1984 and is still an open problem. However, there exist some good bounds for the lengths of matrix sets satisfying some additional conditions. In this talk we discuss the length evaluation problem for quasi-commuting pairs of matrices (we say that A, B in $M_n(\mathbb{F})$ *quasi-commute* if AB and BA are linearly dependent).

First we single out two special classes: (I) commuting pairs and (II) quasi-commutative, non-commuting pairs with a nilpotent product. We show that in each of these cases $l(\mathcal{S}) \leq n-1$ and, moreover, for any $l = 1, \ldots, n-1$, each of these two classes contains a pair of matrices with length l.

If a quasi-commuting pair $\mathcal{S} = \{A, B\} \subset M_n(\mathbb{F})$ does not belong to the class (I)∪(II), then $AB = \varepsilon BA$ where the commutativity factor ε is a primitive k-th root of unity for some $k \leq n$. We will show that in this case the situation is very different from the commutative and nilpotent case. We provide sharp upper and lower bounds for the length of such pairs depending on n, k and the algebraic multiplicity of 0 as an eigenvalue of AB. We show how the interval between these extremal values is divided into intervals of realizable values for the length and “gaps”, i.e. non-realizable values.

This is a joint work with ALEXANDER GUTERMAN (Lomonosov Moscow State University, Russia) and VOLKER MEHRMANN (Technische Universität Berlin, Germany). The talk is partially based on our papers [3, 4].

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Jordan triple product homomorphisms on triangular matrices to and from dimension one

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A map $\Phi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{F})$ is a Jordan triple product (J.T.P.) homomorphism whenever $\Phi(ABA) = \Phi(A)\Phi(B)\Phi(A)$ for all $A, B \in \mathcal{M}_n(\mathbb{F})$.

In work in progress, we study J.T.P. homomorphisms on upper triangular matrices $\mathcal{T}_n(\mathbb{F})$. We characterize J.T.P. homomorphisms $\Phi : \mathcal{T}_n(\mathbb{C}) \to \mathbb{C}$ and J.T.P. homomorphisms $\Phi : \mathbb{F} \to \mathcal{T}_n(\mathbb{F})$ for $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. In the later case we consider continuous maps and the implications of omitting the assumption of continuity.

This is a joint work with DAMJANA KOKOL BUKOVŠEK (University of Ljubljana).

On matrix theory, graph theory, and finite geometry

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Preserver problems represent a research area in matrix theory, where a typical problem demands a characterizations of all maps on a certain set of matrices that preserve some function, subset or a relation. If the studied maps are bijective by the assumption, then the characterization of the maps involved is often easier to obtain. In the case of certain preservers of binary relations it turns out that bijectivity can be deduced automatically by using some techniques from graph theory, which involve graph homomorphisms. In the talk I will survey few such techniques. While several preserver problems on matrices over finite fields have been solved by using these techniques, an increasing number of recent examples shows that both research areas: (a) preserver problems and (b) the study of graph homomorphisms overlap also with some problems in finite geometry.

Inequalities on the spectral radius, operator norm and numerical radius of Hadamard weighted geometric mean of positive kernel operators

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In the talk several inequalities, on the spectral radius ρ, operator norm $\|\cdot\|$ and numerical radius of Hadamard products and ordinary products of non-negative matrices that define operators on sequence spaces, or of Hadamard geometric mean and ordinary products of positive kernel operators on Banach function spaces, will be presented. These inequalities generalize or refine ealier results of several authors. In particular, we show that for a Hadamard geometric mean $A^{\left(\frac{1}{2}\right)} \circ B^{\left(\frac{1}{2}\right)}$ of positive kernel operators A and B on a Banach function space L, we have

$$\rho\left(A^{(\frac{1}{2})} \circ B^{(\frac{1}{2})}\right) \le \rho\left((AB)^{(\frac{1}{2})} \circ (BA)^{(\frac{1}{2})}\right)^{\frac{1}{2}} \le \rho(AB)^{\frac{1}{2}}.$$

In the special case $L = L^2(X, \mu)$ we also prove that

$$\|A^{(\frac{1}{2})} \circ B^{(\frac{1}{2})}\| \le \rho\left((A^*B)^{(\frac{1}{2})} \circ (B^*A)^{(\frac{1}{2})}\right)^{\frac{1}{2}} \le \rho(A^*B)^{\frac{1}{2}}.$$

If time allows, we will also present some related inequalities for the generalized and joint spectral radius for bounded sets of positive kernel operators. The talk is mostly based on the preprints [1, 2].

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Minimal determinantal representations of bivariate polynomials

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It is known since Dixon's 1902 paper that every bivariate polynomial of degree n admits a determinantal representation with $n \times n$ symmetric matrices. However, the construction of such matrices is far from trivial and up to now there have been no efficient numerical algorithms, even if we do not insist on matrices being symmetric. We present a numerical construction of determinantal representations that returns $n \times n$ matrices for a square-free bivariate polynomial of degree n, which, with the exception of the symmetry, agrees with Dixon's result. For a non square-free polynomial one can combine it with a square-free factorization to obtain a representation of order n.

Our motivation is a novel numerical method for solving systems of bivariate polynomials as two-parameter eigenvalue problems. Symmetry is not important for this particular application.

The Birkhoff–James and Roberts orthogonality in C^*-algebras

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Orthogonality in normed linear spaces can be defined in different ways. In this talk, we consider two types of orthogonality in C^*-algebras. The Birkhoff–James orthogonality $a \perp_{BJ} b$ for arbitrary elements a and b of a C^*-algebra $\mathcal{A}$ is characterized in terms of states acting on $\mathcal{A}$. A characterization of a special case of the Roberts orthogonality $a \perp_R e$, where e is the unit in $\mathcal{A}$, is obtained in terms of the Davis–Wielandt shell of a.

This is a joint work with LJILJANA ARAMBAŠIĆ (University of Zagreb) and TOMISLAV BERIĆ (University of Zagreb).

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Dimension of commuting varieties

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Let $C_r(\mathfrak{g})$ denote the set of all r-tuples of commuting elements of the Lie algebra $\mathfrak{g}$ of a linear algebraic group G, and $C_r(\mathcal{N})$ the subset of $C_r(\mathfrak{g})$ consisting of all r-tuples of commuting nilpotent elements. Both sets have natural structures of affine varieties. If $\mathfrak{g}$ is reductive, then $C_2(\mathfrak{g})$ is known to be irreducible and of dimension $\dim \mathfrak{g} + \operatorname{rank} \mathfrak{g}$, while $C_2(\mathcal{N})$ is equidimensional of dimension $\dim[G, G]$. On the other hand, for $r > 2$ these varieties are reducible, except for some small ranks of $\mathfrak{g}$, and very little is known about the irreducible components. We compute the dimension of $C_r(\mathfrak{g})$ and of $C_r(\mathcal{N})$ for sufficiently large r if $\mathfrak{g}$ is of type A or C and the characteristic of the ground field is neither 2 nor 3.

This is a joint work with PAUL D. LEVY (Lancaster University) and NHAM V. NGO (University of Arizona).

A variation principle for ground spaces

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A variation formula is presented for the ground space projections of a vector space of energy operators in a matrix *-algebra. We prove that the ground space projections are the greatest projections of the algebra under certain operator cone constraints. The formula is derived from lattice isomorphisms between normal cones and exposed faces of the state space of the algebra, and between ground space projections.

The vector space of local Hamiltonians is in the focus of quantum many-body physics. The variation formula will be demonstrated with two-local three-bit (commutative) Hamiltonians. A future goal is to understand the lattice of ground spaces of two-local three-qubit (non-commutative) Hamiltonians. Both the combinatorics and topology of this lattice are unsettled issues.

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There are many more positive maps than completely positive maps

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A linear map Φ between matrix spaces is positive if it maps positive semidefinite matrices to positive semidefinite ones, and is called completely positive if all its ampliations $I_n \otimes \Phi$ are positive. We establish quantitative bounds on the fraction of positive maps that are completely positive. A main tool is the real algebraic geometry techniques developed by Blekherman to study the gap between positive polynomials and sums of squares. We also develop an algorithm to produce positive maps which are not completely positive.

This is a joint work with IGOR KLEP (The University of Auckland), SCOTT MCCULLOUGH (University of Florida) and KLEMEN ŠIVIC (University of Ljubljana).

Working groups

Positive maps

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A linear map $\Phi\colon M_n(\mathbb{R}) \to M_m(\mathbb{R})$ is called $*$-*linear* if $\Phi(X^T) = \Phi(X)^T$ for each $X \in M_n(\mathbb{R})$. A $*$-linear map $\Phi\colon M_n(\mathbb{R}) \to M_m(\mathbb{R})$ is called *positive* if it maps positive semidefinite matrices to positive semidefinite matrices. Note that a $*$-linear map on the full matrix space is positive if and only if its restriction to the subspace of symmetric matrices, $\Phi\colon \mathrm{Sym}_n \to \mathrm{Sym}_m$, is positive. For any positive integer k, a $*$-linear map $\Phi\colon M_n(\mathbb{R}) \to M_m(\mathbb{R})$ induces a $*$-linear map $\Phi_k\colon M_{kn} \to M_{km}$ defined by

$$\Phi_k([X_{ij}]_{i,j=1}^k) = [\Phi(X_{ij})]_{i,j=1}^k.$$

The linear map Φ is called *k-positive* if Φ_k is positive. Φ is called *completely positive* if it is k-positive for each positive integer k.

The space of linear maps $\Phi\colon \mathrm{Sym}_n \to \mathrm{Sym}_m$ is isomorphic to the space of biquadratic forms in $n+m$ variables via the isomorphism

$$\Phi \mapsto p_\Phi(\mathtt{x},\mathtt{y}) = \mathtt{y}^T\Phi(\mathtt{x}\mathtt{x}^T)\mathtt{y}.$$

Additionally, Φ is positive if and only the polynomial p_Φ is nonnegative on $\mathbb{R}^n \times \mathbb{R}^m$, and Φ is completely positive if and only if p_Φ is a sum of squares of bilinear forms. Investigating the difference between the convex cones of positive and completely positive maps is therefore the same as investigating the difference between the convex cones of nonnegative and SOS biquadratic forms. The two cones are known to be equal only if $m = 2$ or $n = 2$. However, only few examples of positive maps that are not completely positive are known, see e.g., [2, 3, 5, 6, 7]. There is also an algorithm for constructing such maps in [4], but the positive maps obtained from that algorithm are generically not extremal, i.e., they can be written as a sum of two positive maps.

The aim of the working group is to consider some of the following problems:

- Construct new types of extremal positive maps. In particular, in dimension 3 the above mentioned examples are all of the form

$$\begin{bmatrix} x & y & z \\ y & w & v \\ z & v & u \end{bmatrix} \mapsto \begin{bmatrix} a_1x + a_2w + a_3u & a_4y & a_5z \\ a_4y & a_6x + a_7w + a_8u & a_9v \\ a_5z & a_9v & a_{10}x + a_{11}w + a_{12}u \end{bmatrix}.$$

We would like to know whether in dimension 3 all extremal positive maps that are not completely positive are of that form.

- Quarez [6] initiated the study of the number and configuration of real zeros of nonnegative biquadratic forms with finitely many real zeros. In the case $m = n = 3$ the maximum number of real zeros is 10 [1], but the possible configurations are not known. Moreover, the examples in [1] are the only known nonnegative biquadratic forms with 10 real zeros. Can we construct more examples? Also, there are no examples of nonnegative biquadratic forms with many real zeros if $m > 3$ or $n > 3$.
- If $\Phi\colon \operatorname{Sym}_n \to \operatorname{Sym}_m$ is a positive map, then $\det \Phi(\mathtt{x}\mathtt{x}^{\mathtt{T}}) \geq 0$ for each $\mathtt{x} \in \mathbb{R}^{\mathtt{n}}$, i.e., $\det \Phi(\mathtt{x}\mathtt{x}^{\mathtt{T}})$ is a nonnegative polynomial. We would like to know if all nonnegative polynomials are of this form, and if not, what are the obstructions. In particular, the polynomials $\det\left(\frac{1}{(t^2-1)^{\frac{2}{3}}}\Phi_t(\mathtt{x}\mathtt{x}^{\mathtt{T}})\right)$ constructed in [1] converge to the Robinson's polynomial
$$x_1^6 + x_2^6 + x_3^6 - x_1^4x_2^2 - x_1^4x_3^2 - x_2^4x_1^2 - x_2^4x_3^2 - x_3^4x_1^2 - x_3^4x_2^2 + 3x_1^2x_2^2x_3^2,$$
but the maps $\left(\frac{1}{(t^2-1)^{\frac{2}{3}}}\Phi_t\right)$ do not converge. Can the Robinson's polynomial still be written as $\det \Phi(\mathtt{x}\mathtt{x}^{\mathtt{T}})$ for some positive map Φ? What about the Motzkin's polynomial
$$x_1^4x_2^2 + x_1^2x_2^4 + x_3^6 - 3x_1^2x_2^2x_3^2?$$

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Koopman semigroups

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Already in the 1930s, B. O. Koopman, G. D. Birkhoff, and J. von Neumann observed that a system of nonlinear ordinary differential equations of the form

$$\begin{cases} \dot{x}(t) = F(x(t)), & t \geq 0, \\ x(0) = x_0 \in \mathbb{R}^d, \end{cases} \tag{1}$$

gives rise to a linear one by considering another state space. If the system is well-posed, we can associate a semiflow φ to the system (1) by taking $x(t) = \varphi(t, x_0)$ and derive the corresponding semigroup of Koopman operators, the so-called Koopman semigroup, as

$$(T(t)f)(x) := f(\varphi(t, x)). \tag{2}$$

The new state space is hence a linear space of functions defined on $\mathbb{R}^d$, typically a Banach space, and in this way we obtain a one-parameter semigroup $(T(t))_{t\geq 0}$ of linear operators. As it turns out, many properties of the solutions to (1) can be deduced from the appropriate properties of the associated Koopman semigroup or its generator (which is a derivation, i.e., $A(f \cdot g) = f \cdot Ag + Af \cdot g$).

In the working group we shall discuss possible choices of the new state space (concrete Banach spaces or just locally convex spaces with the appropriate topology) and possible properties of the solutions that can be read off from the corresponding Koopman semigroup or its generator. We shall also consider the case when the semiflow (dynamical system) is defined on a topological space X (i.e., not necessarily comes from an ODE). A particular emphasis will be given to case of non-(locally)-compact X, which is important if the semiflow comes from a PDE, and X is (a convex subset of) an infinite dimensional Banach space.

The aim of the working group is not much to solve concrete problems, but to find ones, and most of all we want to initiate discussion on this topic.

Local-to-global properties for matrix semigroups and quasidiagonal operators and other stuff

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1. If we know a local property for a multiplicative semigroup of matrices, what can we say about the semigroup? For example, let $\mathcal{S}$ be a semigroup of $n \times n$ complex matrices and let P be an orthogonal projection matrix. We may know something about the collection

$$P\mathcal{S}P = \{PAP : A \in \mathcal{S}\},$$

say, it is commutative. Does this imply something about the structure of $\mathcal{S}$? If $P\mathcal{S}P$ is finite or bounded, it is already known that so is $\mathcal{S}$ itself. If we assume commutativity for $P\mathcal{S}P$ (in which case we assume, of course, that the rank of P is more than one) we can show that $\mathcal{S}$ is reducible, i.e., it has a common invariant subspace. What else can we say about the structure? There are other variations of the local commutativity question.

The case where P has rank 1 has been studied by several authors in recent years, and some of the talks given during the conference will address this, and may generate new questions.

2. There are known results of the general form that certain approximate equalities yield the corresponding exact equality. For example, a result of Bernik and Radjavi (2005) says that if in group of unitary $n \times n$ matrices if

$$\|AB - BA\| < \sqrt{3}$$

for all A and B in the group, then the group is commutative. Or, a result of Marcoux, Mastnak and Radjavi (2007) says, among other things, that if A and B are two invertible $n \times n$ matrices such that

$$\operatorname{tr}(A^k - B^k) < 1$$

for all integers k, then A and B have the same spectra (with the same multiplicities). Is it possible to weaken the assumption by restricting k to a bounded subset of integers in this statement? The short answer is no, because you can take A and B close to the identity. But we are looking for long answers: what if you assume something about the norms of A and B?

3. Let M be a nonnegative matrix (i.e., all its entries are nonnegative). If the diagonal of M consists exactly of its eigenvalues with the right multiplicities, then M is triangular after a similarity by a permutation. This was extended to infinite-dimensional setting by Bernik, Marcoux and Radjavi (2012). What about general operators?not necessarily nonnegative? Again we are looking for long answers!

4. A result in Livshits, MacDonald and Radjavi (2011) is the following: let $\mathcal{S}$ be a semigroup of nonnegative matrices (in the sense of Section 3 above) and

assume $\mathcal{S}$ is indecomposable (that is, it has no simultaneous nontrivial invariant subspace spanned by the standard basis vectors). If the diagonals of all members of $\mathcal{S}$ belong to $\{0,1\}$, then after a simultaneous diagonal similarity all the entries of all members of $\mathcal{S}$ are in $\{0,1\}$. What happens if we replace the set $\{0,1\}$ in the hypotheses by another set with a structure?

5. We haven't given the next problem much thought at all, and it might be very easy. (This is not to say that we have thought about all of the above problems that deeply...)

   Suppose that $\mathcal{T} = \mathcal{T}_n(\mathbb{C}) \subseteq \mathbb{M}_n(\mathbb{C})$ is the algebra of upper triangular matrices. The annihilator of $\mathcal{T}$, namely $\mathcal{T}^\perp := \{X \in \mathbb{M}_n(\mathbb{C}) : \mathrm{tr}(T^*X) = 0\}$ is then the set of all *strictly* lower-triangular matrices. Observe that this means that $\mathcal{T}^\perp$ is itself an *algebra*.

   (a) For which algebras $\mathcal{A} \subseteq \mathbb{M}_n(\mathbb{C})$ is it also the case that $\mathcal{A}^\perp$ is again an algebra? In particular, must $\mathcal{A}$ be a finite-dimensional "nest algebra" (i.e. the full set of block-upper triangular matrices with respect to some basis of the Hilbert space)?

   (b) Is there an intrinsic characterization of those subspaces $\mathcal{L} \subseteq \mathbb{M}_n(\mathbb{C})$ which are the annihilators of *some* $\mathrm{Alg}(T)$, the algebra generated by a fixed $T \in \mathbb{M}_n(\mathbb{C})$? For example, a necessary condition is that such a space $\mathcal{L}$ must have dimension at least equal to $n^2 - n$, as $\dim \mathrm{Alg}(T) \leq n$ for all $T \in \mathbb{M}_n(\mathbb{C})$.

6. SPECHT'S THEOREM IN A C^*-ALGEBRA? Suppose that $\mathcal{A}$ is a simple C^*-algebra with a unique tracial state τ and that $a, b \in \mathcal{A}$ satisfy

$$\tau(p(a,a^*)) = \tau(p(b,b^*))$$

   for all polynomials $p(x,y)$ in two non-commuting variables x and y. Is a *approximately unitarily equivalent* to b? That is, does there exist a sequence $(u_n)_n$ of unitary elements of $\mathcal{A}$ so that $b = \lim_n u_n^* a u_n$?

   A test case for this problem would be the case where $\mathcal{A}$ is a *uniformly hyperfinite* (i.e. a (UHF)) C^*-algebra.

Three problems on quasidiagonality

Here are three problems that I (Laurent) have thought about on and off over the years. I am offering these problems up to a wider audience (you or your students) in the hope that I will learn the answer to these questions before I retire (which I expect to do before my esteemed colleague, Heydar Radjavi).

Let $\mathcal{H}$ be an infinite-dimensional, separable, complex Hilbert space. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be **block-diagonal** (we write $T \in (BD)$) if there exists a bounded sequence $(T_n)_n$ of matrices [each $T_n \in \mathbb{M}_{k_n}(\mathbb{C})$ for some $k_n \geq 1$] such that T is unitarily equivalent to $\oplus_{n=1}^\infty T_n$.

An operator $Q \in \mathcal{B}(\mathcal{H})$ is said to be **quasidiagonal** (we write $T \in (QD)$) if it satisfies any one of the three following (equivalent) conditions:

(a) $Q \in \overline{(BD)}$; i.e. Q is a limit of block-diagonal operators;
(b) $Q = B + K$ for some $B \in (BD)$ and $K \in \mathcal{B}(\mathcal{H})$ a compact operator;
(c) Given $\varepsilon > 0$ there exist $B_\varepsilon \in (BD)$ and K_ε a compact operator with $\|K_\varepsilon\| < \varepsilon$ so that $T = B_\varepsilon + K_\varepsilon$.

A. Suppose that $Q \in (QD)$ and that Q is **quasinilpotent** - i.e. the spectrum $\sigma(Q) = \{0\}$. Is Q the limit of block-diagonal nilpotent operators? It is known that it suffices to consider the case where Q is itself block-diagonal (and quasinilpotent).

It is important to note that the approximating nilpotent block-diagonal operators need not be block-diagonal with respect to the same decomposition of the Hilbert space that block-diagonalizes Q.

KNOWN FACTS:

- $Q \oplus T$ is a limit of block-diagonal nilpotent operators in $\mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ whenever $T \in \mathcal{B}(\mathcal{H})$ is a limit of block-diagonal nilpotent operators. In particular, $Q \oplus 0$ is a limit of block-diagonal nilpotent operators.
- If $N \in \mathcal{B}(\mathcal{H})$ is a normal operator, then N is a limit of block-diagonal nilpotent operators if and only if $\sigma(N)$ is connected and contains 0.

One approach to this problem is to try to solve the following matrix problem: let $T \in \mathcal{B}(\mathbb{C}^n)$ be a norm-one operator and $k \geq 1$. If $\varepsilon := \|T^k\|^{1/k}$, find the distance from T to the set of nilpotent matrices in $\mathcal{B}(\mathbb{C}^n)$ in terms of ε and n. In fact, thanks to a result of T. Loring on the lifting of nilpotent elements in quotients of C^*-algebras, the estimate can be made independent of n.

B. Let $(DSN) = \{\oplus_n M_n \in (BD) : \text{ each } M_n \text{ is nilpotent}\}$, so that $D \in (DSN)$ precisely if D is (unitarily equivalent to) a direct sum of (a bounded sequence of) nilpotent matrices. Note that the order of nilpotence of M_n depends upon n, so that D need not be nilpotent itself - e.g. $D = \oplus_n J_n$ where J_n is the $n \times n$ Jordan cell has spectrum equal to the closed unit disk centred at the origin in $\mathbb{C}$.

Let $(ZTR) = \{\oplus_n Z_n \in (BD) : \text{tr}(Z_n) = 0 \text{ for all } n \geq 1\}$, so that $Z \in (ZTR)$ if and only if Z is (unitarily equivalent to) a direct sum of (a bounded sequence of) matrices, each of whose trace is zero.

Let $(BDN) = \{B \in (BD) : \text{there exists } k \geq 1 \text{ so that } B^k = 0\}$. Thus $B \simeq \oplus_n B_n$, and there exists $k \geq 1$ so that $B_n^k = 0$ for all $n \geq 1$.

It is routine to verify that

$$(BDN) \subseteq (DSN) \subseteq (ZTR).$$

Question: is $(ZTR) \subseteq \overline{(BDN)}$? (The *real* question is to characterize $\overline{(BDN)}$.)

C. It is routine to verify the following:
(a) If $A, B \in (QD)$, then $A \oplus B \in (QD)$. More generally, if $X_n \in (QD)$ for all $n \geq 1$ and $\sup_n \|X_n\| < \infty$, then $\oplus_n X_n \in (QD)$.
(b) If $T \in (QD)$ and $K \in \mathcal{B}(\mathcal{H})$ is a compact operator, then $T + K \in (QD)$.

It is also true (but not as routine) that

(c) If $N \in \mathcal{B}(\mathcal{H})$ is a normal operator, then (by the Weyl-von Neumann-Berg Theorem), $N = D + K$ for some diagonalizable operator D and some compact operator K. By (b) above and the fact that a diagonal operator is trivially block-diagonal, $N \in (QD)$.

(d) Suppose that $E \in \mathcal{B}(\mathcal{H})$ is **essentially normal** (i.e. $E^*E - EE^*$ is a compact operator), and that $\|E\| \leq 1$. If $N \in \mathcal{B}(\mathcal{H})$ is a normal operator with $\sigma(N) = \{z \in \mathbb{C} : |z| \leq 1\}$, then (by the Brown-Douglas-Fillmore Theorem), $E \oplus N = M + K$ for some normal operator M and some compact operator K, so that $E \oplus N \in (QD)$.

Suppose that $T = Q \oplus E$ has norm equal to 1, and that Q is quasidiagonal and E is essentially normal. Suppose that $N \in \mathcal{B}(\mathcal{H})$ is a normal operator with $\sigma(N) = \{z \in \mathbb{C} : |z| \leq 1\}$.

From (d) above, $T \oplus N = Q \oplus (E \oplus N) = Q \oplus (M + K)$ is a direct sum of two quasidiagonal operators, and hence T is quasidiagonal.

Is the converse true? That is, suppose that $T \in \mathcal{B}(\mathcal{H})$, $\|T\| \leq 1$, and $T \oplus N$ is quasidiagonal, where N is the normal operator above. Must T be a (compact perturbation) of an operator of the form $Q \oplus E$, where Q is quasidiagonal and E is essentially normal?

The answer is known to be "yes" if T is a weighted shift operator.