Open Linear Maps and Geometry of the Numerical Range

Stephan Weis
Max-Planck-Institute for Mathematics, Leipzig, Germany

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Max Planck Institute for Mathematics in the Sciences

Motivation

- 1877 Ludwig Boltzmann shows that Gibbs distributions maximize *Shannon entropy* under energy constraints
- 1927 John von Neumann shows that quantum Gibbs states maximize *von Neumann entropy* entropy under energy constraints
- 1957 Edwin Jaynes proposes the entropy maximization as a universal *statistical inference method*
- Today the maximum-entropy inference is a recognized method in quantum state reconstruction (Bužek et al. 1999)
- The maximum-entropy inference under linear constraints is always continuous for probability distributions on a finite space but can be discontinuous for quantum states (W. & Knauf 2012)
- Open problem for 3×3-matrices: What is the structure of the (dis-) continuity of the maximum-entropy inference?

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Linear Constraints on a Quantum State Space

- $M_d(\mathbb{F})$ the full matrix algebra of $d \times d$ matrices, $d \in \mathbb{N}$, over the field $\mathbb{F}$ of complex numbers $\mathbb{C}$ or real numbers $\mathbb{R}$; identity matrix $\mathbb{1}_d \in M_d(\mathbb{F})$, zero matrix $0_d \in M_d(\mathbb{F})$
- $\mathcal{A} \subset M_d(\mathbb{C})$ a *real* C*-subalgebra; in quantum mechanics take a *complex* C*-subalgebra $\mathcal{A}$
- Euclidean space $\mathcal{A}_{\mathrm{sa}} := \{a \in \mathcal{A} \mid a^* = a\}$, $\langle a, b \rangle := \mathrm{tr}(ab)$; in quantum mechanics $\mathcal{A}_{\mathrm{sa}}$ is the space of *observables*
- *state space* $\mathcal{M} := \{\rho \in \mathcal{A} \mid \rho \succeq 0, \mathrm{tr}(\rho) = 1\}$, positive semi-definite matrices of trace one called *density matrices* or *states*; in quantum mechanics $\langle \rho, a \rangle \in \mathbb{R}$ is the *expected value* of the observable $a \in \mathcal{A}_{\mathrm{sa}}$ if the system is in the state $\rho \in \mathcal{M}$
- we fix observables $u_1, \ldots, u_k \in \mathcal{A}_{\mathrm{sa}}$, $k \in \mathbb{N}$, and define the *expected value functional* $\mathbb{E} : \mathcal{A}_{\mathrm{sa}} \to \mathbb{R}^k$, $a \mapsto (\langle a, u_1 \rangle, \ldots, \langle a, u_k \rangle)$
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Examples of State Spaces $\mathcal{M} = \mathcal{M}(\mathcal{A})$

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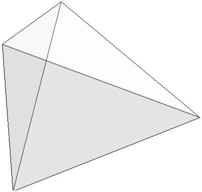
$\mathcal{A} \cong \mathbb{C}^4$

simplex



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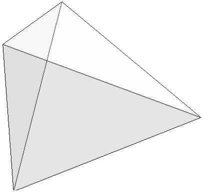
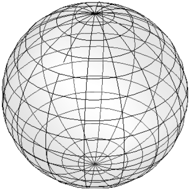


$\mathcal{A} = M_2(\mathbb{C})$

Bloch ball



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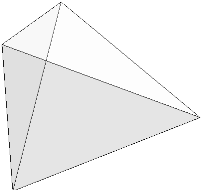

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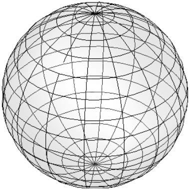

$\mathcal{A} = M_3(\mathbb{C})$

$\mathcal{M} =$?

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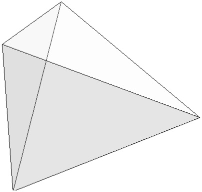

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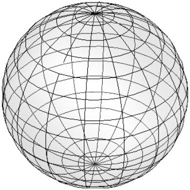

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For $d \geq 3$ the state space $\mathcal{M}(M_d(\mathbb{C}))$ is neither a polytope nor a ball. But the direct sum algebra $\mathcal{D} := M_2(\mathbb{R}) \oplus \mathbb{R}$ of real matrices $\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{pmatrix}$ suffices to illustrate the continuity problem.

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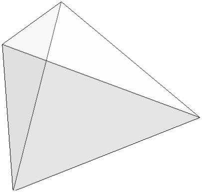

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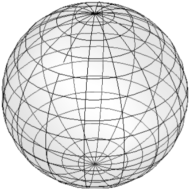

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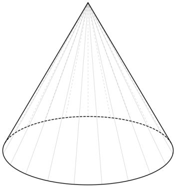
$\mathcal{A} = \mathcal{D}$

cone



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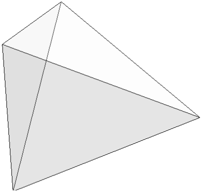

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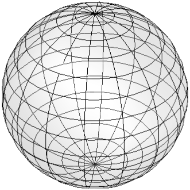

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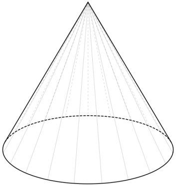
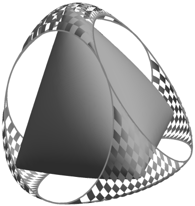


For curiosity:
$\mathcal{M}(M_3(\mathbb{C})) \cap A$ inside the orthogonal projection of $\mathcal{M}(M_3(\mathbb{C}))$ to the affine space A of real matrices $\begin{pmatrix} 1/3 & x & y \\ x & 1/3 & z \\ y & z & 1/3 \end{pmatrix}$



The Maximum-Entropy Inference

- *von Neumann entropy* $S : \mathcal{M} \to \mathbb{R}$, $\rho \mapsto -\operatorname{tr} \rho \log(\rho)$,
  S is continuous and strictly concave, S has a unique maximum on every convex body included in $\mathcal{M}$

- *maximum-entropy inference* $\Psi : \mathcal{C} \to \mathcal{M}$,
  $x \mapsto \operatorname{argmax}\{S(\rho) \mid \rho \in \mathbb{E}|_{\mathcal{M}}^{-1}(x)\}$

- we call $\mathbb{E}|_{\mathcal{M}} : \mathcal{M} \to \mathcal{C}$ *open* at $\rho \in \mathcal{M}$ if $\mathbb{E}(N)$ is a neighborhood of $\mathbb{E}(\rho)$ for every neighborhood $N \subset \mathcal{M}$ of ρ;
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Proof. For all $N \subset \mathcal{M}$ we have $\Psi^{-1}(N) = \mathbb{E}(N \cap \Psi(\mathcal{C})) \subset \mathbb{E}(N)$. So continuity implies openness.

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If $x \in \mathcal{C}$ then Ψ is continuous at x if and only if $\mathbb{E}|_{\mathcal{M}}$ is open at $\Psi(x)$.

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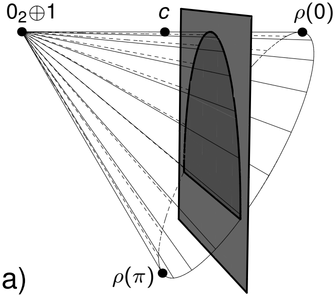

Fig. a). The cone $\mathcal{M}$. The subset, denoted N, left from the depicted plane is a neighborhood of $c := \frac{1}{2}(0_2 \oplus 1 + \rho(0))$

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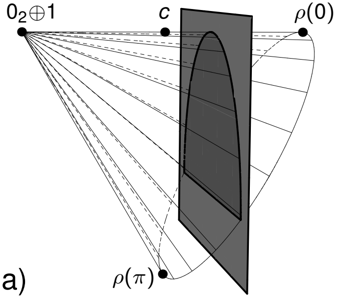

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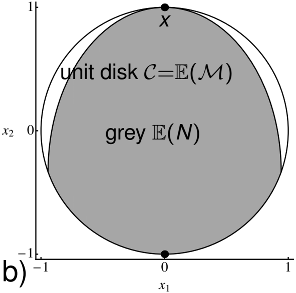

Figure b). The set of expected values $\mathcal{C} = \mathbb{E}(\mathcal{M})$ for $u_1 := \sigma_1 \oplus 0$ and $u_2 := \sigma_3 \oplus 1$, where $\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are *Pauli* matrices. The image $\mathbb{E}(N)$ is bounded by an ellipse of curvature > 1 near x, so $\mathbb{E}(N)$ is not a neighborhood of $x := (0,1) = \mathbb{E}(0_2 \oplus 1) = \mathbb{E}(\rho(0))$.

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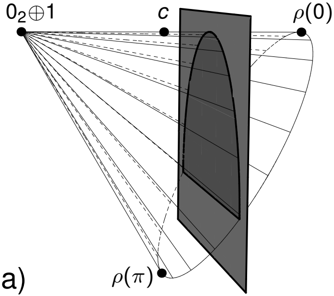

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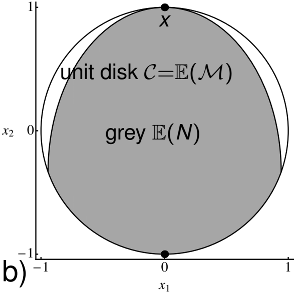

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Sufficient Openness Conditions

Let $(X, \|\cdot\|)$ be a real normed vector space, $C \subset X$ a convex subset, $x \in C$. We define, respectively, the *ball* and *sphere* of radius $\epsilon > 0$

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The *gauge* of C is the function $\gamma_C : X \to [0,\infty]$,

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Proposition (W. 2014)

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Lemma (One-point fibers)

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The sufficient conditions allow us to discuss the cone $\mathcal{M} = \mathcal{M}(\mathcal{D})$.

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If $\mathcal{A} = \mathcal{D}$ then $\mathbb{E}|_{\mathcal{M}}$ is open on $\mathcal{M}$ unless $\dim(\mathcal{C}) = 2$ and a generatrix of $\mathcal{M}$ is a fiber of $\mathbb{E}|_{\mathcal{M}}$, that is $[\rho \oplus 0, 0_2 \oplus 1] = \mathbb{E}|_{\mathcal{M}}^{-1} \circ \mathbb{E}(0_2 \oplus 1)$ holds for some $\rho \in M_2(\mathbb{R})$. In the latter case $\mathbb{E}|_{\mathcal{M}}$ is not open at any point of $]\rho \oplus 0, 0_2 \oplus 1]$ and open on the complement.

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Using rotation in the first summand of $\mathcal{D} = M_2(\mathbb{R}) \oplus \mathbb{R}$ (double cover $SO(2) \to SO(2)$ restricted from $SU(2) \to S(3)$), we have the following.

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Discussion: A Grassmannian Perspective

- Invertible linear transformations preserve openness. For two linearly independent observables u_1, u_2 we replace $\mathbb{E}$ by the orthogonal projection onto a plance $V \subset \mathcal{D}_{\mathrm{sa}}$ of trace-less matrices, $\dim(V) = 2$, that is an element of the Grassmannian $G_{2,3}$. We identify $\mathcal{C} = \mathbb{E}(\mathcal{M})$ and the image of $\mathcal{M}$ on V.

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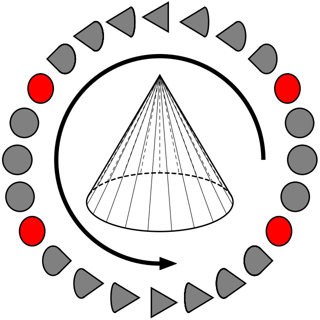
Figure:
Orthogonal projections of $\mathcal{M}(\mathcal{D})$ to planes $V \in G_{2,3}$. Planes V with a discontinuous maximum-entropy inference are marked by a red $\mathcal{C}(V)$.



Discussion: Geometry of $\mathcal{C}$ and Numerical Range

A discontinuity exists for $V \in G_{2,3}$ where

- flat boundary portions of $\mathcal{C}(V)$ disappear,
- non-exposed points of $\mathcal{C}(V)$ disappear.

A *non-exposed* point of $\mathcal{C}$ (green) is an extreme point which is not the unique maximizer in $\mathcal{C}$ of a linear functional.

$\mathcal{C}$ is equivalent to the *numerical range* of
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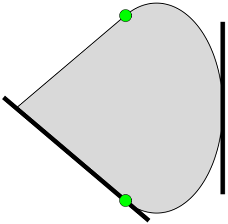

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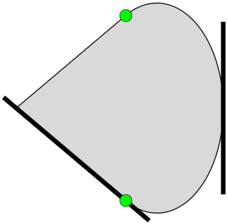

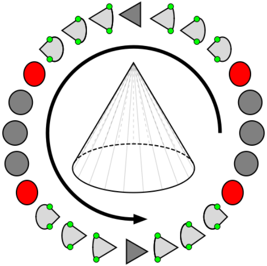

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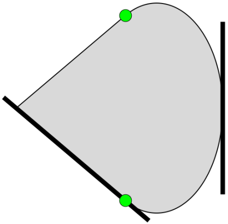

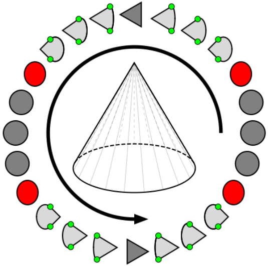

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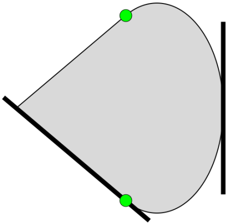

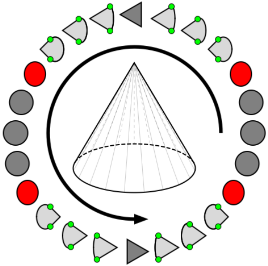

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Do geometric results about the numerical range help in the continuity analysis of Ψ in $M_3(\mathbb{C})$?

Thanks!

Credits:

Supported by the DFG (German Research Foundation) project (10/2011–09/2014)
Quantum Statistics: Decision problems and entropic functionals on state spaces

Proof. (Proposition on Slide 7)

Let $\epsilon > 0$ and denote the diameter of $\mathcal{M}$ by $d := \max_{\tau_1,\tau_2 \in \mathcal{M}} \|\tau_2 - \tau_1\|$. If $d = 0$ then nothing is to prove so let $d > 0$. For $\rho \in \mathbb{E}|_{\mathcal{M}}^{-1}(x)$ we have

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