Multinorms and Banach lattices

Based on results of G.Dales, M.Polyakov, N.Laustsen, G.Pisier, L.McClaran, P.Ramsden, T.Oikhberg

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The only multinorm on $\mathbb{R}$ is the ℓ_∞-norm.

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The only 1-multinorm on $\mathbb{R}$ is the ℓ_1-norm.

Theorem

A sequence of norms is a multinorm iff
$\|A\bar{x}\|_m \leqslant \|A\|\|\bar{x}\|_n$ *for every* $\bar{x} \in X^n$ *and every* $A \in M_{m,n}$;
where $(A\bar{x})_i = \sum_{j=1}^{n} a_{ij}x_j$ *and* $\|A\| = \|A\colon \ell_\infty^n \to \ell_\infty^m\|$

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p-multinorms satisfy (A1), (A2), and (A3).

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$$\left\|\left(x_1 + Y, \ldots, x_n + Y\right)\right\| := \inf_{y_1, \ldots, y_n \in Y} \left\|(x_1 + y_1, \ldots, x_n + y_n)\right\|$$

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This is a q-multinorm on X^*, where $q = p^*$.

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X and Y are **multiisometric** if there is a surjective multiisometry from X onto Y.

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Conversely, a norm on $c_{00} \otimes X$ induces a sequence of norms on X^n. When is this sequence a p-multinorm?

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$n^{-\frac{1}{q}} \sum_{i=1}^{n} |x_i| \leqslant \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \leqslant \sum_{i=1}^{n} |x_i|$, where $q = p^*$.

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Hence T is p-multibounded and $\|T\|_{p-\mathrm{mb}} = \|T\|$.

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The converse is false in general.

Theorem

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What happens for other values of p?

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It follows also that $\|T\|_{2-\mathrm{mb}} \leqslant K_G \|T\|$.

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That is, for every multinormed space X there exists a Banach lattice E and a linear map $T\colon X \to E$ such that

$$\left\|(x_1, \ldots, x_n)\right\| = \left\|\bigvee_{i=1}^{n} |Tx_i|\right\|$$

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That is, there is a one-to-one correspondence between multinorms on X and embeddings of X into Banach lattices as a subspace.

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1-multinormed spaces = quotients of Banach lattices

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Theorem

Every convex strong p-multinormed space is multiisometric to a subspace of a Banach lattice with the canonical p-multinorm.

Without extra assumptions, there are examples of p-multinormed spaces which cannot be embedded into a Banach lattice with the canonical p-multinorm (as a subspace).

Convex p-multinorms

A p-multinorm is **convex** if

$$\left\|(x_1, \dots, x_n)\right\|^p \leqslant \left\|(x_1, \dots, x_k)\right\|^p + \left\|(x_{k+1}, \dots, x_n)\right\|^p$$

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If $p=1$, trivial.

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Fact: The canonical p-multinorm on a Banach lattice is strong.

When $p = 2$ or $p = \infty$, every p-multinorm is strong

Proof.

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Let V be the order ideal generated by $j(X)$ in $C(K)$,

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Put $E = \overline{V/\ker\rho}$. Put $T\colon X \to E$ via $Tx = \hat{x} + \ker\rho$. Then

$$\Big\|\Big(\sum_{i=1}^{n} |Tx_i|^p\Big)^{\frac{1}{p}}\Big\| = \big\|(x_1, \dots, x_n)\big\|.$$