Identities on matrices

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Polynomial identities

Definition

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Polynomial identities

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An element $f \in \mathbb{F}\langle x_1, \ldots, x_d\rangle$ is a *polynomial identity* of an algebra A if

$$f(a_1, \ldots, a_d) = 0 \quad \text{for all } a_1, \ldots, a_d \in A.$$

Polynomial identities

$M_n(\mathbb{F})$

Standard polynomial: $S_d(x_1, \ldots, x_d) = \sum_{\sigma \in S_d} (-1)^\sigma x_{\sigma(1)} \cdots x_{\sigma(d)}$.

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Amitsur-Levitzki 1950

S_{2n} is a polynomial identity of $M_n(\mathbb{F})$.

Polynomial identities

Specht problem

An ideal I in $\mathbb{F}\langle x_1, \ldots, x_d \rangle$ is a T- *ideal* if $f(g_1, \ldots, g_d) \in I$ for all $g_1, \ldots, g_d \in \mathbb{F}\langle x_1, \ldots, x_d \rangle$, $f \in I$.

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Kemer 1987

T-ideal of polynomial identities of $M_n(\mathbb{F})$ is finitely generated.

Trace identities
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$$x^2 - \mathrm{tr}(x)x + \det(x)1 = 0 \text{ on } M_2(\mathbb{F}).$$

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A *trace polynomial* is a noncommutative polynomial that involves also traces, e.g. $-2x_1x_2x_3 + \mathrm{tr}(x_1^2)\mathrm{tr}(x_2x_3)x_4 + \mathrm{tr}(x_1)\mathrm{tr}(x_2)\mathrm{tr}(x_3)1$.

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A trace polynomial f is a *trace identity* of $M_n(\mathbb{F})$ if

$$f(a_1, \dots, a_d) = 0 \quad \text{for all } a_1, \dots, a_d \in M_n(\mathbb{F}).$$

Trace identities

Description

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Procesi 1976, Razmyslov 1974
Every trace identity is a “consequence” of the Cayley-Hamilton identity.

Quasi-identities

Definition

A *quasi-identity* of $M_n(\mathbb{F})$ is an identity of the form

$$\sum_M \lambda_M(x_1, \ldots, x_d) M,$$

where $M \in \mathbb{F}\langle x_1, \ldots, x_d \rangle$ is a noncommutative monomial and $\lambda_M : M_n(\mathbb{F})^d \to \mathbb{F}$ is a polynomial function.

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- λ_M induced by traces $\rightsquigarrow$ trace identity

Quasi-identities

Example

Antisymmetrization of

$$x_{12}^{(1)}\,x_{21}^{(2)}\left(X_3 - \mathrm{tr}(X_3)\right)\left(X_4 - \mathrm{tr}(X_4)\right)$$

vanishes on $M_2(\mathbb{F})$.

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Brešar, Procesi, Š. 2014

No.

Quasi-identities

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An element $f \in \mathbb{F}\langle x_1, \ldots, x_d \rangle$ is a *central polynomial* of $M_n(\mathbb{F})$ if $f(a_1, \ldots, a_d) \in \mathbb{F}1$ for all $a_1, \ldots, a_d \in M_n(\mathbb{F})$.

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Brešar, Procesi, Š. 2014

Let P be a quasi-identity of $M_n(\mathbb{F})$. For every central polynomial c of $M_n(\mathbb{F})$ with zero constant term there exists $m \in \mathbb{N}$ such that $c^m P$ is a consequence of the Cayley-Hamilton identity.

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A T-ideal of quasi-identities is finitely generated.

Functional identities

Definition

A *functional identity* is an identity of the form

$$\sum_{k \in K} F_k(\overline{x}_d^k) x_k = \sum_{l \in L} x_l G_l(\overline{x}_d^l).$$

Functional identities

Examples

- $\big[Q_2(ax, by), cz\big] = 0$ on $M_2(\mathbb{F})$.
- $xF_1(y) + yF_2(x) = G_1(y)x + G_2(x)y$ has a *standard solution*

$$\begin{aligned} F_1(y) &= ay + \lambda(y), \\ F_2(x) &= bx + \mu(x), \\ G_1(y) &= yb + \lambda(y), \\ G_2(x) &= xa + \mu(x). \end{aligned}$$

Functional identities

Description

Brešar, Š., 2014

Every one-sided functional identity of $M_n(\mathbb{F})$ is a consequence of the Cayley-Hamilton identity.

Every solution of a functional identity on $M_n(\mathbb{F})$ is standard modulo one-sided identities.