Patterns of symmetric matrices that allow all the eigenvalue multiplicities to be even

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Graph

A (simple undirected) graph $G = (V(G), E(G))$ consists of:

- vertices $V(G) = \{1, 2, \ldots, n\}$,
- edges $E(G)$

An edge is two element subset of $V(G)$.

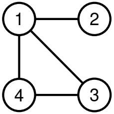
$\mathcal{S}(G)$

$\mathcal{S}_n$: the set of all symmetric matrices in $M_n(\mathbb{R})$

$\mathcal{S}(G) = \{A \in S_{|G|};\ \text{ for } i \neq j,\ a_{ij} \neq 0 \text{ if and only if } \{i, j\} \in E(G)\}.$

(No conditions on the diagonal entries of $A \in S(G)$.)

Example



$$A = \begin{pmatrix} 1 & -2 & 0.7 & 11 \\ -2 & 0 & 0 & 0 \\ 0.7 & 0 & -3 & -0.3 \\ 11 & 0 & -0.3 & 0 \end{pmatrix} \in \mathcal{S}(G)$$

The inverse eigenvalue problem

Let $G = (V(G), E(G))$ be a graph on n vertices.

- When is a list of real numbers $\{\lambda_1, \ldots, \lambda_n\}$ the spectrum of a matrix $A \in \mathcal{S}(G)$?
- What are possible multiplicity lists for the eigenvalues of $A \in \mathcal{S}(G)$?
- What is the maximum possible multiplicity of an eigenvalue λ of $A \in \mathcal{S}(G)$?

Minimum Rank Problem

The minimum rank of a graph G is:

$$\mathrm{mr}(G) = \min\{\mathrm{rank}(A);\ A \in \mathcal{S}(G)\}.$$

Maximum possible multiplicity $= |G| - \mathrm{mr}(G)$

Bibliography

There exists extensive literature on minimum rank problem.

Fallat, Hogben (2007) and Fallat, Hogben (2013): two survey papers with more than 100 further recent references on the problem.

Even multiplicities

For which graphs G does there exists $A \in S(G)$ with all its eigenvalue multiplicities even?

- The multiplicities of all the eigenvalues of A are even.
- $\det(xI_n - A) = q(x)^2,$ where $q(x) \in \mathbb{R}[x]$
- $\det(xI_n - A) \geq 0$ for all $x \in \mathbb{R}$.

G allows a square characteristic polynomial, if there exists $A \in S(G)$ satisfying conditions above.

Trees

Johnson, Leal Duarte (2002): Let G be a tree, then the largest and the smallest eigenvalue of any matrix $A \in \mathcal{S}(G)$ has multiplicity 1.

Let G be a tree, then a matrix $A \in \mathcal{S}(G)$ doesn't have the characteristic polynomial a square.

Minimum number of distinct eigenvalues

q(G)- minimum number of distinct eigenvalues of a graph G.

Ahmadi, Alinaghipour, Cavers, Fallat, Meagher, Nasserasr (2013): If there are vertices u, v in a connected graph G at distance d and the path of length d from u to v is unique, then $q(G) \geq d + 1$.

G, $|G| = 2n$, allows square characteristic polynomial, then $q(G) \leq n$.

Theorem

Let A be a matrix of the form

$$A = \begin{pmatrix} d & b^T & c^T \\ b & B & 0 \\ c & 0 & D \end{pmatrix} \in M_{2n}(\mathbb{R}), \tag{1}$$

where $d \in \mathbb{R}$, $B \in M_2(\mathbb{R})$ and $D \in M_{2n-3}(\mathbb{R})$ diagonal matrix and $b \in \mathbb{R}^2$ and $c \in \mathbb{R}^{2n-3}$ are vectors with all its components nonzero. Then the characteristic polynomial of A is not a square.

Vector Space

Theorem

Let $\mathcal{V}_{2n}$ be a set of symmetric matrices of the form

$$\begin{pmatrix} A & S \\ -S & A \end{pmatrix},$$

such that $A \in M_n(\mathbb{R})$ is a symmetric matrix, and $S \in M_n(\mathbb{R})$ a skew symmetric matrix.

$\mathcal{V}_{2n}$ is a vector space in which the characteristic polynomial of every matrix is a square.

Proof

The characteristic polynomial of $M = \begin{pmatrix} A & S \\ -S & A \end{pmatrix}$ is a square.

$$\begin{pmatrix} I_n & iI_n \\ 0 & I_n \end{pmatrix} \begin{pmatrix} A & S \\ -S & A \end{pmatrix} \begin{pmatrix} I_n & -iI_n \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} A + iS & 0 \\ -S & A - iS \end{pmatrix}.$$

Since $(A + iS)^T = A - iS$, the spectrum of $A + iS$ is equal to the spectrum of $A - iS$.

Cycles

Corollary

There exists $M \in \mathcal{S}(C_{2n})$, such that the characteristic polynomial of M is a square.

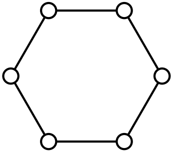
Proof.

$M = \begin{pmatrix} A & S \\ -S & A \end{pmatrix}$, where

$$S = E_{n1} - E_{1n} \in M_n(\mathbb{R}) \text{ and } A = \sum_{|i-j|=1} E_{i,j} \in M_n(\mathbb{R})$$

□

C_6



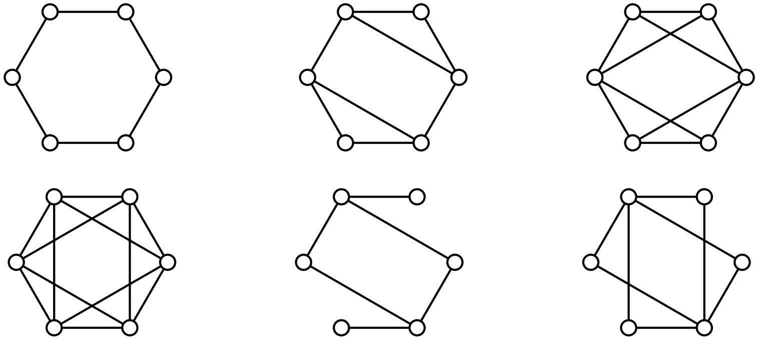
$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \; S = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$M = \begin{pmatrix} A & S \\ -S & A \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\det(x\, I_6 - M) = x^2(x^2 - 3)^2.$$

Example

Graphs on 6 vertices that are covered by vectors space V_6:



Products of graphs

- G: a graph on n vertices with the adjacency matrix A.
- H: a graph on m vertices with the adjacency matrix B.

Tensor product of graphs

$G \times H = (V(G \times H), E(G \times H))$:

- $V(G \times H) = V(G) \times V(H)$
- $((u, u'), (v, v')) \in E(G \times H) \Leftrightarrow (u, u') \in E(G)$ and $(v, v') \in E(H)$

The adjacency matrix of $G \times H$: $A \otimes B$

Kronecker product $A \otimes B$ has eigenvalues

$$\lambda_i \mu_j,\ i = 1, \ldots, n,\ j = 1, \ldots, m$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A and $\mu, \ldots, \mu_m$ are the eigenvalues of B.

Tensor product

Let $A \in S_{2n}(G)$ have the characteristic polynomial a square. Then $A \otimes B$ has the characteristic polynomial a square for any $B \in \mathcal{S}_m$.

We do not control the pattern of the diagonal elements of $A \in \mathcal{S}(G)$ with characteristic polynomial a square, but we can make them all nonzero by adding a scalar.

The Cartesian product of graphs

$G\square H = (V(G\square H), E(G\square H))$:

- $V(G\square H) = V(G) \times V(H)$
- $((u_1, u_2), (v_1, v_2)) \in E(G\square H)$ if and only if either $u_1 = v_1$ and $(u_2, v_2) \in E(H)$ or $u_2 = v_2$ and $(u_1, v_1) \in E(G)$.

The adjacency matrix of $G\square H$: $(I_m \otimes A) + (B \otimes I_n)$ (the *Kronecker sum* of A and B)

$(I_m \otimes A) + (B \otimes I_n)$ has eigenvalues $\lambda_i + \mu_j$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A \in M_n(\mathbb{R})$ and $\mu_1, \mu_2, \ldots, \mu_m$ are the eigenvalues of $B \in M_m(\mathbb{R})$.

Theorem

Let G be a graph that allows a square characteristic polynomial. Then the Cartesian product $G \square H$ allows a square characteristic polynomial for any graph H.

The strong product $G \boxtimes H$ of graphs

$G \boxtimes H = (V(G \boxtimes H), E(G \boxtimes H))$:

- $V(G \boxtimes H) = V(G) \times V(H)$
- $((u_1, u_2), (v_1, v_2)) \in E(G \boxtimes H)$ if and only if either $u_1 = v_1$ and $(u_2, v_2) \in E(H)$ or $u_2 = v_2$ and $(u_1, v_1) \in E(G)$ or $(u_1, v_1) \in E(G)$ and $(u_2, v_2) \in E(H)$.

The adjacency matrix of $G \boxtimes H$: $((A + I_n) \otimes (B + I_m)) - I_{mn}$.

The matrix $((A + I_n) \otimes (B + I_m)) - I_{mn}$ has eigenvalues $(\lambda_i + 1)(\mu_j + 1) - 1$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A \in M_n(\mathbb{R})$ and $\mu_1, \mu_2, \ldots, \mu_m$ are the eigenvalues of $B \in M_m(\mathbb{R})$.

Theorem

Let G be a graph that allows square characteristic polynomial. Then the strong product $G \boxtimes H$ allows square characteristic polynomial for any graph H.

Construction

Lemma

Let B be a symmetric $m \times m$ matrix with eigenvalues $\mu_1, \mu_2, \ldots, \mu_m$, and let u an eigenvector corresponding to μ_1 normalized so that $u^T u = 1$. Let A be an $n \times n$ symmetric matrix with a diagonal element μ_1 :

$$A = \begin{pmatrix} A_1 & b \\ b^T & \mu_1 \end{pmatrix}. \tag{2}$$

and eigenvalues $\lambda_1, \ldots, \lambda_n$. Then the matrix

$$C = \begin{pmatrix} A_1 & bu^T \\ ub^T & B \end{pmatrix}$$

has eigenvalues $\lambda_1, \ldots, \lambda_n, \mu_2, \ldots, \mu_m$.

Theorem

Let G be a graph that allows a square characteristic polynomial. Let v be a vertex in G. Let G_{2m+1} be a graph constructed from G in the following way: vertex v is replaced by a clique K_{2m+1}, and every vertex in K_{2m+1} has the same neighbours in the rest of the graph as v has in G. Then G_{2m+1} can be realized by a matrix whose characteristic polynomial is a square.

$$A = \begin{pmatrix} A_1 & b \\ b^T & 2m+1 \end{pmatrix} \in S(G), \det(xI - A) = p(x)^2. \qquad (3)$$

$$C = \begin{pmatrix} A_1 & be^T \\ eb^T & J \end{pmatrix}, \det(xI - C) = x^{2m} p(x)^2$$

Lemma

Let

$$A = \begin{pmatrix} A_1 & a \\ a^T & 2 \end{pmatrix} \in S_n$$

have the characteristic polynomial $p(x)$, *and let*

$$B = \begin{pmatrix} B_1 & b \\ b^T & 0 \end{pmatrix} \in S_m$$

have the characteristic polynomial $q(x)$. *Then*

$$C = \begin{pmatrix} B_1 & 0 & \frac{\sqrt{2}}{2}b & -\frac{\sqrt{2}}{2}b \\ 0 & A_1 & \frac{\sqrt{2}}{2}a & \frac{\sqrt{2}}{2}a \\ \frac{\sqrt{2}}{2}b^T & \frac{\sqrt{2}}{2}a^T & 1 & 1 \\ -\frac{\sqrt{2}}{2}b^T & \frac{\sqrt{2}}{2}a^T & 1 & 1 \end{pmatrix}$$

has the characteristic polynomial $p(x)q(x)$.

Corollary

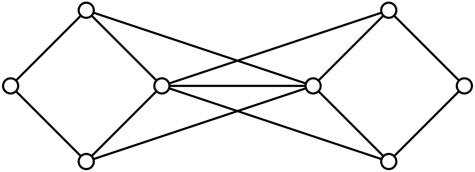
Let G_A and G_B be graphs that allow characteristic polynomial a square. Let $v_A \in V(G_A)$ and $v_B \in V(G_B)$. Let G_C be a graph constructed by adding the following edges to $G_A \cup G_B$:

- $(v_A, v_B) \in E(G_C)$
- *v_A has the same edges to the rest of G_B as v_B*
- *v_B has the same edges to the rest of G_A as v_A.*

Then G allows a square characteristic polynomial.

Example

If $G_1 = G_2 = C_4$, then graph G:



allows a square characteristic polynomial.

Inverse eigenvalue problem of a complete graph

Theorem

For any given list of real numbers $\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_n)$*,* $\lambda_1 \neq \lambda_2$*, there exists* $A_n \in S(K_n)$ *with the spectrum* σ*.*

Furthermore, given any zero-nonzero pattern of a vector in $\mathbb{R}^n$ *that contains at least two nonzero elements,* A_n *can be chosen in such a way that there exist an eigenvector corresponding to* λ_1 *with the given pattern.*

Join with a complete graph

Corollary

For every graph G on $n-1$ vertices there exists $A \in \mathcal{S}(G \vee K_{n+1})$ with the characteristic polynomial a square.

Proof.

- $A_1 \in \mathcal{S}(G)$, $b \in \mathbb{R}^{n-1}$ with only nonzero entries, $\mu_1 \in \mathbb{R}$. $A = \begin{pmatrix} A_1 & b \\ b^T & \mu_1 \end{pmatrix}$. Eigenvalues of A: $\lambda_1, \dots, \lambda_n$.
- Let $B \in \mathcal{S}(K_{n+1})$ have eigenvalues $\mu_1, \lambda_1, \dots, \lambda_n$ and an eigenvector u corresponding to μ_1 with only nonzero entries.
- $\begin{pmatrix} A_1 & bu^T \\ ub^T & B \end{pmatrix} \in \mathcal{S}(A_1 \vee K_{n+1})$ has eigenvalues $\lambda_1, \lambda_1, \dots, \lambda_n, \lambda_n$.

Rank 2

$$\mathrm{mr}_+(G) = \min\{\mathrm{rank}(A);\ A \in \mathcal{S}(G),\ A \text{ positive semidefinite }\}.$$

Barrett, van der Holst, Loewy (2004) A characterisation of graphs with $\mathrm{mr}_+(G) \leq 2$.

Theorem

Let G be a graph on n vertices. Then there exist a matrix $A \in \mathcal{S}(G)$ with the characteristic polynomial $p(x) = x^{n-2}(x-a)^2$ if and only if G^c has the form

$$(K_{p_1,q_1} \cup K_{p_2,q_2} \cup \ldots \cup K_{p_k,q_k}) \vee K_r \tag{4}$$

Example

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \sqrt{2} & \sqrt{2} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{2}{\sqrt{3}} & -\frac{2}{\sqrt{3}} & -\frac{2}{\sqrt{3}} \end{pmatrix}$$

$$U^T U = \begin{pmatrix} * & * & 0 & 0 & * & * & * & * & * \\ * & * & 0 & 0 & * & * & * & * & * \\ 0 & 0 & * & * & * & * & * & * & * \\ 0 & 0 & * & * & * & * & * & * & * \\ * & * & * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 & * & * & * \\ * & * & * & * & 0 & 0 & * & * & * \\ * & * & * & * & 0 & 0 & * & * & * \end{pmatrix}, \; UU^T = \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix}.$$

Example

$$U^T U = \begin{pmatrix} * & * & 0 & 0 & * & * & * & * & * \\ * & * & 0 & 0 & * & * & * & * & * \\ 0 & 0 & * & * & * & * & * & * & * \\ 0 & 0 & * & * & * & * & * & * & * \\ * & * & * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 & * & * & * \\ * & * & * & * & 0 & 0 & * & * & * \\ * & * & * & * & 0 & 0 & * & * & * \end{pmatrix} \in \mathcal{S}(G),$$

where $G^c = K_{2,2} \cup K_{2,3}$.