NUMERICAL RANGES OF QUATERNION MATRICES

Leiba RODMAN

Department of Mathematics
The College of William and Mary
Williamsburg, VA 23187–8795
e-mail: lxrodm@gmail.com

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Notation

$\mathbf{H}$ the skew field of real quaternions

$x = a_0 + a_1\mathsf{i} + a_2\mathsf{j} + a_3\mathsf{k} \in \mathbf{H}, \qquad a_0, a_1, a_2, a_3 \in \mathbf{R}$

Then

$$\mathfrak{R}(x) = a_0,$$

$$\mathfrak{V}(x) = a_1\mathsf{i} + a_2\mathsf{j} + a_3\mathsf{k}$$

$x^* = a_0 - a_1\mathsf{i} - a_2\mathsf{j} - a_3\mathsf{k}$

$|x| = \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}$

$A = [a_{i,j}] \in \mathbf{H}^{m\times n}$, then

$A^* = [a_{j,i}^*] \in \mathbf{H}^{n\times m}$

Numerical ranges with respect to conjugation

The set

$$W_*^{\mathbf{H}}(A) := \left\{x^* A x \,:\, x^* x = 1, \qquad x \in \mathbf{H}^{n\times 1}\right\} \subset \mathbf{H}$$

is known as the (quaternion) *numerical range* of

$$A \in \mathbf{H}^{n\times n}$$

with respect to the conjugation.

Elementary properties:

(1) $W_*^{\mathbf{H}}(A)$ is compact and connected

(2) If $y_1 \in W_*^{\mathbf{H}}(A)$ and $y_2 \in \mathbf{H}$ is such that $\mathfrak{R} y_2 = \mathfrak{R} y_1$ and $|\mathfrak{V} y_2| = |\mathfrak{V} y_1|$, then also $y_2 \in W_*^{\mathbf{H}}(A)$

(3) $W_*^{\mathbf{H}}(A) = \{0\}$ if and only if $A = 0$

(4) $W_*^{\mathbf{H}}(A) \subset \mathbf{R}$ if and only if $A = A^*$

(5) $\mathfrak{R}(W_*^{\mathbf{H}}(A)) = \{0\}$ if and only if $A = -A^*$

(6) for $A \in \mathbf{H}^{n\times n}$, unitary $U \in \mathbf{H}^{n\times n}$, and real a, we have

$$W_*^{\mathbf{H}}(U^*AU) = W_*^{\mathbf{H}}(A),$$

$$W_*^{\mathbf{H}}(A + aI) = a + W_*^{\mathbf{H}}(A),$$

$$W_*^{\mathbf{H}}(aA) = aW_*^{\mathbf{H}}(A).$$

The quaternion numerical ranges are generally non-convex:

Example

Let

$$A = \begin{bmatrix} \lambda & 0 \\ 0 & A_0 \end{bmatrix}, \qquad n \geq 2,$$

where $\lambda \in \mathbf{H} \setminus \{0\}$ has zero real part, and $A_0 \in \mathbf{H}^{(n-1)\times(n-1)}$ is hermitian and either positive or negative definite. Clearly, $\lambda \in W_*^{\mathbf{H}}(A)$, and, therefore, also $-\lambda \in W_*^{\mathbf{H}}(A)$ (because $-\lambda$ is congruent to λ). But one easily checks that

$$0 = \frac{1}{2}\lambda + \frac{1}{2}(-\lambda)$$

does not belong to $W_*^{\mathbf{H}}(A)$. Indeed, if we had

$$x^*\lambda x + y^*A_0 y = 0, \quad \text{for some } x \in \mathbf{H}, \; y \in \mathbf{H}^{(n-1)\times 1},$$

then, since $\mathfrak{R}\,(x^*\lambda x) = 0$ and $\mathfrak{V}\,(y^*A_0 y) = 0$, we must have

$$x^*\lambda x = y^*A_0 y = 0,$$

which yields $x = 0$ and $y = 0$.

So, Thompson (1996): Intersection of $W_*^{\mathbf{H}}(A)$ with closed upper complex half plane is convex

Open Problem

Prove (or disprove) that a convex set $S \subseteq \mathbf{H}$ *has the property that* $W_*^{\mathbf{H}}(A) \cap S$ *is convex for every* $A \in \mathbf{H}^{n\times n}$ *if and only if there is no nonreal* $\lambda \in S$ *such that* $\overline{\lambda} \in S$

Open Problem

Identify classes of matrices for which convexity of the numerical range holds true

Joint numerical ranges

For a p-tuple of hermitian matrices

$$A_1, \ldots, A_p \in \mathbf{H}^{n \times n},$$

the $\mathbf{H}$-*joint numerical range* is defined by

$$WJ_*^{\mathbf{H}}(A_1, \ldots, A_p)$$

$$:= \left\{ (x^* A_1 x, \ldots, x^* A_p x) \in \mathbf{R}^p \, : \, x^* x = 1, \quad x \in \mathbf{H}^{n \times 1} \right\}$$

Subset of $\mathbf{R}^p$.

Basic convexity result:

Theorem

(1) *If* $n \neq 2$ *and* $A_1, \ldots, A_5 \in \mathbf{H}^{n \times n}$ *are hermitian, then* $WJ_*^{\mathbf{H}}(A_1, \ldots, A_5)$ *is convex.*

(2) *If* $A_1, \ldots, A_4 \in \mathbf{H}^{n \times n}$ *are hermitian, then*

$$WJ_*^{\mathbf{H}}(A_1, \ldots, A_4)$$

is convex.

(3) *Let* $A_1, \ldots, A_5 \in \mathbf{H}^{2\times 2}$ *be hermitian. Then*

$$WJ_*^{\mathbf{H}}(A_1, \ldots, A_5)$$

is convex if and only if the 6*-tuple of matrices* $\{A_1, \ldots, A_5, I_2\}$ *is linearly dependent over the reals.*

Parts (1) and (2): Au-Yeung, Poon (1979)

<u>Global vs local geometry of numerical ranges</u>

A subset $\mathbf{D}$ of $\mathbf{R}^k$ will be called a *d-dimensional halfspace* if there exist d linearly independent vectors $v_1, \ldots, v_d \in \mathbf{R}^k$ and a nonzero vector $v_0 \in \text{span}\,\{v_1, \ldots, v_d\}$ such that

$$\mathbf{D} = \{v \in \text{span}\,\{v_1, \ldots, v_d\} \,:\, \langle v, v_0 \rangle \leq 0.\}$$

$\mathbf{F} = \mathbf{R}$, $\mathbf{F} = \mathbf{C}$, or $\mathbf{F} = \mathbf{H}$

Theorem.

The following statements are equivalent for a *pair of hermitian matrices*

$$(A_1, A_2), \qquad A_j \in \mathbf{F}^{n\times n}.$$

(a) $WJ_*^{\mathbf{F}}(A_1, A_2)$ *is contained in a* 2*-dimensional half-space.*

(b) $WJ_*^{\mathbf{F}}(X^*A_1X, X^*A_2X)$ *is contained in*

a 2*-dimensional halfspace*

for every isometry X *into* $\mathbf{F}^n$.

(c) *There is* $m \geq 2$ *such that* $WJ_*^{\mathbf{F}}(X^*A_1X, X^*A_2X)$ *is contained in*

a 2*-dimensional halfspace*

for every isometry $X : \mathbf{F}^m \longrightarrow \mathbf{F}^n$.

(d) $WJ_*^{\mathbf{F}}(X^*A_1X, X^*A_2X)$ *is contained in*

a 2*-dimensional halfspace*

for every isometry $X : \mathbf{F}^2 \longrightarrow \mathbf{F}^n$.

Holds also for selfadjoint operators in infinite dimensional Hilbert space

Cheung, Li, R. (2007), R. (2008)

Extend to k-tuples:

Conjecture

The following statements are equivalent for a k-tuple of $n \times n$ hermitian matrices $A = (A_1, \ldots, A_k)$, and for a fixed integer d, $0 \leq d \leq k$:

(a) $WJ_*^{\mathbf{F}}(A)$ *is contained in a d-dimensional halfspace.*

(b) $WJ_*^{\mathbf{F}}(X^*AX)$ *is contained in a d-dimensional halfspace for every isometry X into $\mathbf{F}^n$.*

(c) *There is $m \geq d$ such that $WJ_*^{\mathbf{F}}(X^*AX)$ is contained in a 2-dimensional halfspace for every isometry $X : \mathbf{F}^m \longrightarrow \mathbf{F}^n$.*

(d) $WJ_*^{\mathbf{F}}(X^*AX)$ *is contained in a d-dimensional halfspace for every isometry $X : \mathbf{F}^d \longrightarrow \mathbf{F}^n$.*

Numerical ranges: other involutions

A map $\phi : \mathbf{H} \longrightarrow \mathbf{H}$ is called an *antiendomorphism* if

$$\phi(xy) = \phi(y)\phi(x) \quad \forall \quad x, y \in \mathbf{H},$$

and

$$\phi(x + y) = \phi(x) + \phi(y) \quad \forall \quad x, y \in \mathbf{H}.$$

An antiendomorphism ϕ is called an *involution* if

$$\phi(\phi(x)) = x \quad \forall \quad x \in \mathbf{H}.$$

Ex.: $\phi(x) = x^*$

Other examples:

$$\phi(a_0 + a_1\mathsf{i} + a_2\mathsf{j} + a_3\mathsf{k}) = a_0 + a_1\mathsf{i} + a_2\mathsf{j} - a_3\mathsf{k}$$

In general: If ϕ is an involution different from the conjugation, then ϕ is real linear and

$$\phi = \begin{bmatrix} 1 & 0 \\ 0 & T \end{bmatrix}$$

with respect to the real basis $\{1, \mathsf{i}, \mathsf{j}, \mathsf{k}\}$, where T is 3×3 real orthogonal symmetric matrix with eigenvalues $1, 1, -1$

There is $v \in \mathbf{H}$, $|v| = 1$, unique up to negation, such that $\phi(v) = -v$

$$\mathrm{Inv}\,(\phi) = \{x \in \mathbf{H} \,:\, \phi(x) = x\}$$

3-dimensional real subspace of $\mathbf{H}$

$A = [a_{i,j}]_{i=1,j=1}^{m,n} \in \mathbf{H}^{m\times n}$ then

$$A_\phi = [\phi(a_{j,i}] \in \mathbf{H}^{n\times m}$$

Numerical ranges

$A \in \mathbf{H}^{n \times n}$

fixed $\alpha \in \mathrm{Inv}\,(\phi)$:

$$W_\phi^{(\alpha)}(A) := \{x_\phi A x \,:\, x_\phi x = \alpha, \quad x \in \mathbf{H}^{n\times 1}\}.$$

Writing $\alpha = \gamma_\phi \gamma$ for some $\gamma \in \mathbf{H}$ we see that

$$W_\phi^{(\alpha)}(A) = \gamma_\phi W_\phi^{(1)}(A)\gamma,$$

assuming $\alpha \neq 0$. Thus, we can focus on

$$W_\phi^{(1)}(A) \quad \text{and} \quad W_\phi^{(0)}(A).$$

To avoid trivialities, in the latter case $n \geq 2$ will be assumed.

Elementary properties:

Proposition *If* $A, U \in \mathbf{H}^{n\times n}$, *where* U *is* ϕ*-unitary, and if* $a \in \mathbf{R}$, *then*

$$W_\phi^{(\alpha)}(U_\phi A U) = W_\phi^{(\alpha)}(A),$$

$$W_\phi^{(\alpha)}(A + aI) = W_\phi^{(\alpha)}(A) + a\alpha,$$

$$W_\phi^{(\alpha)}(aA) = a\,W_\phi^{(\alpha)}(A)$$

for every $\alpha \in \mathrm{Inv}\,(\phi)$.

$W_\phi^{(\alpha)}(A)$ is connected but not necessarily bounded

Open Problem *Identify those* $\alpha \in \mathrm{Inv}\,(\phi)$ *and* $A \in \mathbf{H}^{n\times n}$ *for which* $W_\phi^{(\alpha)}(A)$ *is bounded.*

A complete answer is available for the case $\alpha = 0$:

Theorem

The numerical range $W_\phi^{(0)}(A)$, *where* $A \in \mathbf{H}^{n\times n}$, $n \geq 2$, *is bounded if and only if* $n \geq 3$ *and* $A = aI$ *for some real* a, *or* $n = 2$ *and* A *has the form*

$$A = \begin{bmatrix} a_0 + a_1\beta & a_2 + a_3\beta \\ -a_2 + a_3\beta & a_0 - a_1\beta \end{bmatrix}$$

for some $a_0, a_1, a_2, a_3 \in \mathbf{R}$

Proposition

Let $A \in \mathbf{H}^{n\times n}$. Then:

(1) $W_\phi^{(\alpha)}(A) = \{0\}$ *for some (equivalently, for all)* $\alpha \in \mathrm{Inv}\,(\phi) \setminus \{0\}$ *if and only if* $A = 0$*;*

(2) $W_\phi^{(\alpha)}(A) \subseteq \mathrm{Inv}\,(\phi)$ *for some (equivalently, for all)* $\alpha \in \mathrm{Inv}\,(\phi) \setminus \{0\}$ *if and only if* $A = A_\phi$*;*

(3) $W_\phi^{(\alpha)}(A)$ *is contained in the real span of* β *for some (equivalently, for all)* $\alpha \in \mathrm{Inv}\,(\phi) \setminus \{0\}$ *if and only if* $A = -A_\phi$*.*

This result is false for $\alpha = 0$:

Example

Let $\beta \in \mathbf{H}$ be such that $|\beta| = 1$ and $\phi(\beta) = -\beta$.
Let $A = \begin{bmatrix} \beta & 0 \\ 0 & -\beta \end{bmatrix}$.
We claim that

$$W_\phi^{(0)}(A) = \{0\}.$$

Indeed, for $x = \begin{bmatrix} b \\ c \end{bmatrix} \in \mathbf{H}^{2\times 1}$, where $b, c \in \mathbf{H}$, the condition $x_\phi x = 0$ amounts to $b_\phi b + c_\phi c = 0$, which, in turn, implies $|b| = |c|$ (because ϕ is an isometry on $\mathbf{H}$). On the other hand, we have $b_\phi \beta b = \beta |b|^2$; hence,

$$x_\phi A x = b_\phi \beta b - c_\phi \beta c = \beta(|b|^2 - |c|^2),$$

which is equal to zero as long as $|b| = |c|$.

Theorem.

Let $A \in \mathbf{H}^{n\times n}$, $n \geq 2$.

(1) $W_\phi^{(0)}(A) = \{0\}$ *if and only if either* $n = 2$ *and* A *has the form*

$$A = \begin{bmatrix} a_0 + a_1\beta & a_2 + a_3\beta \\ -a_2 + a_3\beta & a_0 - a_1\beta \end{bmatrix}$$

for some $a_0, a_1, a_2, a_3 \in \mathbf{R}$, *or* $n \geq 3$ *and* $A = aI$ *for some real* a.

(2) $W_\phi^{(0)}(A) \subseteq \mathrm{Inv}\,(\phi)$ *if and only if* $n \geq 3$ *and* A *is* ϕ*-hermitian* ($A_\phi = A$) *or* $n = 2$ *and* A *has the form*

$$A = \begin{bmatrix} a_1\beta & a_2 + a_3\beta \\ -a_2 + a_3\beta & -a_1\beta \end{bmatrix} + B,$$

for some $a_1, a_2, a_3 \in \mathbf{R}$ *and some* ϕ*-hermitian matrix* B.

(3) $W_\phi^{(0)}(A)$ *is contained in the real span of* β *if and only if* A *has the form* $A = aI + B$, *where* $a \in \mathbf{R}$ *and* B *is* ϕ*-skewhermitian.*

Joint numerical ranges: other involutions

Fix an involution ϕ other than conjugation

Fix $\alpha \in \mathrm{Inv}\,(\phi)$

For a p-tuple of ϕ-hermitian matrices

$$A_1, \ldots, A_p \in \mathbf{H}^{n\times n},$$

let

$$WJ_{\phi}^{(\alpha)}\,(A_1, \ldots, A_p)$$

$$:= \{(x_{\phi}A_1x, \ldots, x_{\phi}A_px) \,:\, x_{\phi}x = \alpha, \quad x \in \mathbf{H}^{n\times 1}\},$$

be the *joint ϕ-numerical range* of $A_1, \ldots, A_p$.

$$WJ_{\phi}^{(\alpha)}\,(A_1, \ldots, A_p) \subseteq (\mathrm{Inv}\,(\phi))^p$$

Open Problem.

Study geometric properties of joint ϕ-numerical ranges versus algebraic properties of the constituent matrices.

For ϕ-skewhermitian matrices: $A_\phi = -A$

another version of joint numerical ranges:

p-tuple of ϕ-skewhermitian $n \times n$ quaternion matrices $(A_1, \ldots, A_p)$

Define the *joint ϕ-numerical range*

$$WJ_\phi(A_1, \ldots, A_p) := \{(x_\phi A_1 x, x_\phi A_2 x, \ldots, x_\phi A_p x) : x \in \mathbf{H}^n, \ \ \|x\| = 1\} \subseteq \mathbf{H}^p.$$

Since $\phi(x_\phi A x) = -x_\phi A x$, we clearly have that

$$WJ_\phi(A_1, \ldots, A_p) \subseteq \{(y_1\beta), \ldots, y_p\beta) : y_1, y_2, \ldots, y_p \in \mathbf{R}\}.$$

Theorem

(1) *If* $n \neq 2$*, then*

$$WJ_\phi(A_1, A_2, A_3, A_4, A_5)$$

is convex for every 5*-tuple of* ϕ*-skewhermitian matrices* $A_1, \ldots, A_5$.

(2) *If* $n = 2$*, then* $WJ_\phi(A_1, A_2, A_3, A_4)$ *is convex for every* 4*-tuple of* ϕ*-skewhermitian matrices* $A_1, \ldots, A_4$.

(3) *If* $n = 2$*, then* $WJ_\phi(A_1, A_2, A_3, A_4, A_5)$ *is convex for a* 5*-tuple of* ϕ*-skewhermitian matrices* $A_1, \ldots, A_5$ *if and only if the* 6*-tuple of* ϕ*-skewhermitian matrices* $(A_1, \ldots, A_5, \beta I)$ *is linearly dependent over the reals.*

Numerical range in a halfplane

We identify here $\mathbf{R}^2\beta$ with $\mathbf{R}^2$.

Theorem.

The following statements are equivalent for a pair of ϕ-skewhermitian $n \times n$ matrices (A, B):

(1) $WJ_\phi(A,B)$ *is contained in a half-plane bounded by a line passing through the origin;*

(2) *the pencil $A + tB$ is ϕ-congruent to a pencil of the form $\beta A' + t\beta B'$, where A' and B' are real symmetric matrices such that some linear combination $(\sin\mu)A' + (\cos\mu)B'$, $0 \le \mu < 2\pi$ is positive semidefinite.*

ϕ-congruence:

$$A + tB \quad \longrightarrow \quad S_\phi AS + tS_\phi BS,$$

where $S \in \mathbf{H}^{n\times n}$ is invertible.