On the spectrum and the spectral mapping theorem in max algebra

Aljoša Peperko; joint work with Vladimir Müller

* University Ljubljana;

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In the talk:

Results for $n \times n$ non-negative matrices:

- new description of spectrum in max algebra via **local spectral radii**

- related results for the **usual spectrum** of **complex matrices** and **distinguished spectrum** of **non-negative matrices**

- application: a new proof of the **spectral mapping theorem in max algebra**

Results for non-negative bounded infinite matrices A:

- **Bonsall's cone spectral radius** of a map $x \mapsto A \otimes x$ is **included** in its **max algebra approximate point spectrum**

- we investigate the **spectral mapping theorem** with respect to **point** and **approximate point spectrum in max algebra**

- corresponding results are valid for more general **max** and **max-plus** type kernel operators and its **tropical** versions (**Bellman operators**)

Max algebra:

- **algebraic system**, very useful for describing certain conventionally **non-linear problems** in a **linear fashion**

- these problems are appearing for instance in:

  machine-scheduling, discrete event-dynamic systems, combinatorial optimisation, mathematical physics, information technology , DNA analysis, ...

- some recent results in **linear algebra** by **max-algebra techniques**:
  - L. Elsner, P. van den Driessche, LAA 2008: shorter proof of **Al'pin's bounds for the spectral radius** of a nonnegative matrix
  - V. Müller, P., LAA 2013: short proof of the **Berger-Wang formula** for the joint and generalized spectral radius of bounded sets of **nonnegative** matrices
  - M. Sharify, S. Gaubert, Akian, 2008 - ... : by tropical polynomial methods improving the accuracy of the numerical computation of the **eigenvalues of a matrix polynomial**

Notations:

- **max algebra**: semifield $[0, \infty)$
  $a \oplus b = \max\{a, b\}$, **usual product** ab

- $A \geq 0$ ($A \in \mathbb{R}_+^{n \times n}$), if $a_{ij} \geq 0$ for all i, j

- similarly $x \in \mathbb{R}_+^n$

- $a_{ij} \leftrightarrow A_{ij}$

Notations:

- **max sum** $[A \oplus B]_{ij} = \max\{a_{ij}, b_{ij}\}$

  max products $[A \otimes B]_{ij} = \max_{k=1,\dots,n} a_{ik} b_{kj}$,

  $(A \otimes x)_i = \max_{j=1,\dots,n} a_{ij} x_j$

  max powers $A^2_\otimes = A \otimes A$, ..., $A^k_\otimes$

- usual associative and distributive laws hold in this (semi)algebra

- **spectrum** in max algebra $\sigma_{\otimes}(A)$:

  the set of all (max eigenvalues) $\lambda \geq 0$ for which there exists $x \in \mathbb{R}^n_+$, $x \neq 0$ with $A \otimes x = \lambda x$.

- the role of the spectral radius in max algebra:

  $r_{\otimes}(A)$ **maximum cycle geometric mean**

  $r_{\otimes}(A) = \max\{(A_{i_1 i_k} \cdots A_{i_3 i_2} A_{i_2 i_1})^{1/k} : k \in \mathbb{N}$ and $i_1, \ldots, i_k \in \{1, \ldots, n\}\}$

  $r_{\otimes}(A) = \max\{(A_{i_1 i_k} \cdots A_{i_3 i_2} A_{i_2 i_1})^{1/k} : k \leq n$ and $i_1, \ldots, i_k \in \{1, \ldots, n\}$ mutually distinct$\}$

- **max Perron-Frobenius**: $r_{\otimes}(A) = \max\{\lambda : \lambda \in \sigma_{\otimes}(A)\}$

  if A is irreducible, then $r_{\otimes}(A)$ is the **unique** max eigenvalue and every max eigenvector is **positive**

- **Gelfand formula**: $r_{\otimes}(A) = \lim_{k \to \infty} \|A_{\otimes}^{k}\|^{1/k}$

  for any $A \in \mathbb{R}_{+}^{n \times n}$ and for an arbitrary vector norm $\|\cdot\|$ on $\mathbb{R}^{n \times n}$

Description of $\sigma_{\otimes}(A)$ via local spectral radii:

- for $x \in \mathbb{R}^n_+$ denote $\|x\| = \max_{i=1,\ldots,n} x_i$

- **local spectral radius** in max algebra at x:

  $r_x(A) = \limsup_{k\to\infty} \|A^k_{\otimes} \otimes x\|^{1/k}$

- $e_1, \ldots, e_n$ are standard vectors, then $\|A^k_{\otimes} \otimes e_j\|$ is the largest entry of the jth column of the matrix $A^k_{\otimes}$, i.e.,

  $\|A^k_{\otimes} \otimes e_j\| = \max\{A_{j_k,j_{k-1}} \cdots A_{j_2,j_1} A_{j_1,j} : 1 \le j, j_1, \cdots, j_k \le n\}.$

Theorem 1 *If $A \in \mathbb{R}_+^{n\times n}$, then*

$$\sigma_{\otimes}(A) = \{\lambda : \text{ there exists } j \in \{1, \ldots, n\}, \lambda = r_{e_j}(A)\}.$$

Steps in the proof of Theorem 1:

Lemma 2 *Let* $A \in \mathbb{R}_{+}^{n\times n}$, $j \in \{1,\ldots,n\}$. *Then* $r_{e_j}(A)$ *is the maximum of all* $\lambda \geq 0$ *with the following property* (*):

there exist $a \geq 0$, $b \geq 1$ *and mutually distinct indices* $i_0 := j, i_1, \ldots, i_a, i_{a+1}, \ldots, i_{a+b-1} \in \{1,\ldots,n\}$ *such that*

$$\prod_{s=0}^{a-1} A_{i_{s+1},i_s} \neq 0 \quad \text{and} \quad \prod_{s=a}^{a+b-1} A_{i_{s+1},i_s} = \lambda^b,$$

where we set $i_{a+b} = i_a$.

Remark. Lemma 2 states that for each $j \in \{1,\ldots,n\}$, the radius $\lambda = r_{e_j}(A)$ equals the **maximum of cycle geometric means**, such that the **node** j **is accessible** from one of the **corresponding ($\lambda-$critical) cycles.**

Corollary 3 *If* $A \in \mathbb{R}_{+}^{n \times n}$, *then*

$$r_{\otimes}(A) = \max_{j=1,\ldots,n} r_{e_j}(A).$$

Theorem 4 *Let* $A \in \mathbb{R}_{+}^{n \times n}$, $x = (x_1, \ldots, x_n) \in \mathbb{R}_{+}^{n}$, $x \neq 0$. *Then:*

(i) the limit $\lim_{k \to \infty} \|A_{\otimes}^{k} \otimes x\|^{1/k}$ *exists;*

(ii) $r_x(A) = \max\{r_{e_j}(A) : 1 \leq j \leq n, x_j \neq 0\}$.

- set $C \subset \mathbb{R}_+^n$ is *a max cone*, if $x \oplus y \in C$ and $\lambda x \in C$ for all $x, y \in C$ and $\alpha \geq 0$.

- for a set $S \subset \mathbb{R}_+^n$, $\bigvee S$ denotes the max cone generated by S:

  finite max-combinations $x = \alpha_1 s_1 \oplus \cdots \oplus \alpha_{k(x)} s_{k(x)}$, $\alpha_i \geq 0$, $s_i \in S$

- max cone C is *invariant* for $A \in \mathbb{R}_+^{n \times n}$, if $A \otimes x \in C$ for all $x \in C$

Corollary 5 *Let* $A \in \mathbb{R}_+^{n \times n}$, $t \geq 0$. *Then*
$\{x \in \mathbb{R}_+^n : r_x(A) \leq t\} = \bigvee \{e_j : r_{e_j}(A) \leq t\}$
is a max cone invariant for A.

Theorem 1. If $A \in \mathbb{R}_+^{n\times n}$, then

$$\sigma_\otimes(A) = \{\lambda : \text{ there exists } j \in \{1,\ldots,n\}, \lambda = r_{e_j}(A)\}.$$

Proof. If $\lambda \in \sigma_\otimes(A)$, then $\lambda = r_x(A)$ ... x corresponding max-eigenvector. **By Theorem** 4: $\lambda = r_{e_j}(A)$ for some $j \in \{1,\ldots,n\}$.

Conversely, let $j \in \{1,\ldots,n\}$ and $t := r_{e_j}(A)$. Let $M = \bigvee\{e_s : r_{e_s}(A) \le t\}$. **By Corollary 5**, the max cone M is **invariant for** A. For the **restriction** $A|_M$ of A to M we have **by Corollary 3** that $r_\otimes(A|_M) = \max\{r_{e_s}(A|_M) : e_s \in M\} = t \Rightarrow$ there exists $x \in M \subset \mathbb{R}_+^n$, $x \neq 0$ with $A \otimes x = tx$.

Some consequences/remarks:

- we obtain a **block triangular form** of $A \in \mathbb{R}_+^{n\times n}$ related to its **Frobenius normal form** and deduce a known description (S. Gaubert, 91) of $\sigma_\otimes(A)$ via **access relations**

- the usual local spectral radius $\rho_x(A)$ of $A \in \mathbb{C}^{n\times n}$ at $x \in \mathbb{C}^n$: $\rho_x(A) = \limsup_{k\to\infty} \|A^k x\|^{1/k}$,

  With **similar proof:**

  Proposition 6 *If* $A \in \mathbb{C}^{n\times n}$, *then*
  $\{|\lambda| : \lambda \in \sigma(A)\} = \{t :$ *there exists* $x \in \mathbb{C}^n, t = \rho_x(A)\}$.

- distinguished spectrum $\sigma_D(A)$ of $A \in \mathbb{R}_+^{n\times n}$:

$$\sigma_D(A) = \{\lambda \geq 0 : \text{ there exists } x \in \mathbb{R}_+^n, x \neq 0 \text{ such that } Ax = \lambda x\}$$

With **similar proof:**

Theorem 7 *If $A \in \mathbb{R}_+^{n\times n}$, then*

$$\sigma_D(A) = \{\lambda : \textit{ there exists } j \in \{1, \ldots, n\}, \lambda = \rho_{e_j}(A)\}.$$

Application: Spectral mapping theorem in max algebra

$\mathcal{P}_+$ the set of all polynomials with non-negative coefficients, $\mathcal{P}_+ = \{p = \sum_{j=0}^{\deg p} \alpha_j z^j : \alpha_j \geq 0, j = 0, \dots \deg p\}$.

For $p \in \mathcal{P}_+$ and for $t \geq 0$ write $p_\otimes(t) = \max\{\alpha_j t^j : 0 \leq j \leq \deg p\}$ (**polynomial in max algebra**)

For $A \in \mathbb{R}_+^{n \times n}$ define

$$p_\otimes(A) = \bigoplus_{j=0}^{\deg p} \alpha_j A_\otimes^j.$$

For $p, q \in \mathcal{P}_+$, $p = \sum_{j=0}^{\deg p} \alpha_j z^j$, $q = \sum_{j=0}^{\deg q} \beta_j z^j$ define $p \oplus q, p \otimes q \in \mathcal{P}_+$ by

$$p \oplus q = \sum_{j=0}^{\max\{\deg p, \deg q\}} \max\{\alpha_j, \beta_j\} z^j,$$

$$p \otimes q = \sum_{j=0}^{\deg p + \deg q} \max\{\alpha_i \beta_{j-i} : 0 \leq i \leq j\} z^j.$$

Proposition 8 *Let* $p, q \in \mathcal{P}_+$ *and* $A \in \mathbb{R}_+^{n \times n}$. *Then*

$(p \oplus q)_\otimes(A) = p_\otimes(A) \oplus q_\otimes(A)$ *and* $(p \otimes q)_\otimes(A) = p_\otimes(A) \otimes q_\otimes(A)$.

Thus $p \mapsto p_\otimes(A)$ defines naturally a **polynomial functional calculus** for matrices $A \in \mathbb{R}_+^{n \times n}$.

Let $A \in \mathbb{R}_+^{n\times n}$ and $p \in \mathcal{P}_+$. Katz, Schneider, Sergeev, LAA,2012 proved

$$\sigma_\otimes(p_\otimes(A)) = p_\otimes(\sigma_\otimes(A)). \tag{1}$$

by using the existence theorem of **common max eigenvectors for commutative matrices** in max algebra applied to A and $p_\otimes(A)$.

- (1) follows also from **Theorem 1**

- (1) holds also for **max power series** by sufficient **continuity properties** of $\sigma_\otimes(A)$

Infinite non-negative bounded matrices $A \in M_+^{\infty\times\infty}$:
$A = [A_{ij}]_{i,j=1}^{\infty}$, $A_{ij} \geq 0$, $\|A\| := \sup_{i,j\in\mathbb{N}} A_{ij} < \infty$

- for $x, y \in l_+^{\infty}$ we write $\|x - y\| = \sup_{i\in\mathbb{N}} |x_i - y_i|$.

- **in definitions replace:** max by sup

- Since $\|A \otimes x - A \otimes y\| \leq \|A\| \cdot \|x - y\|$ for all $x, y \in l_+^{\infty}$,

  a map $g_A : l_+^{\infty} \to l_+^{\infty}$, $g_A : x \mapsto A \otimes x$ is **well defined continuous max linear** (in particular, monotone, positively homogeneous) map.

- We also have $\|A\| = \sup\{\|A \otimes x\| : \|x\| \leq 1, x \in l_{+}^{\infty}\} = \sup\left\{\frac{\|A\otimes x\|}{\|x\|} : x \neq 0, x \in l_{+}^{\infty}\right\} = \sup_{j\in\mathbb{N}} \|A \otimes e_j\|$.

- $r_{\otimes}(A) := \lim_{k\to\infty} \|A_{\otimes}^{k}\|^{1/k} = \inf_{k\in\mathbb{N}} \|A_{\otimes}^{k}\|^{1/k}$

  equals by definition **Bonsall's cone spectral radius** $r_{l_{+}^{\infty}}(g_A)$ of the map g_A with respect to the cone l_{+}^{∞} (g_A is monotone, positively homogeneous and continuous)

- local spectral radius at x: $r_x(A) = \limsup_{k\to\infty} \|A_{\otimes}^{k} \otimes x\|^{1/k}$

  $r_x(A) \leq r_{\otimes}(A)$ for all $x \in l_{+}^{\infty}$. Moreover, $r_y(A) = r_{\otimes}(A)$ for $y = (1, 1, \ldots)$.

Let $\{e_j : j \in \mathbb{N}\}$ be standard vectors.

In general $\sup r_{e_j}(A) \neq r_\otimes(A)$ and Theorem 1 for **infinite matrices is not true.**

Example 9 S the left shift on l_+^∞, i.e., $S \otimes e_1 = 0$ and $S \otimes e_j = e_{j-1}$ for all $j \geq 2$. Then $r_{e_j}(S) = 0$ for each e_j, but $r_\otimes(S) = 1 \in [0, 1] = \sigma_p(S) = \sigma_{ap}(S)$.

- **point spectrum** $\sigma_p(A)$ in max algebra of $A \in M_+^{\infty \times \infty}$: the set of all $\lambda \geq 0$ for which there exists $x \in l_+^\infty$, $\|x\| = 1$ with $A \otimes x = \lambda x$.

- **approximate point spectrum** $\sigma_{ap}(A)$ in max algebra

- **approximate point spectrum** $\sigma_{ap}(A)$ is the set of all $\lambda \geq 0$ for which there exists a sequence $(x_k) \subset l^\infty_+$, $\|x_n\| = 1$ such that $\lim_{k\to\infty} \|A \otimes x_k - tx_k\| = 0$.

- $\sigma_p(A) \subset \sigma_{ap}(A)$, $\sigma_{ap}(A)$ is always **closed** and **nonempty**, since $r_\otimes(A) \in \sigma_{ap}(A)$

- Example: S left shift; for its restriction $S|_{c_0}$ to c_0 we have $\sigma_p(S|_{c_0}) = [0, 1)$ and $\sigma_{ap}(S|_{c_0}) = [0, 1]$.

Mallet-Paret, Nussbaum 2002 proved that **under certain compactness assumptions** it holds

$r_{\otimes}(A) = \max\{t : t \in \sigma_p(A)\}$.

Theorem 10 *Let* $A \in M_{+}^{\infty\times\infty}$. *Let* $\sup\{r_{e_j}(A) : j \in \mathbb{N}\} \leq t \leq r_{\otimes}(A)$. *Then* $t \in \sigma_{ap}(A)$.
In particular, $r_{\otimes}(A) \in \sigma_{ap}(A)$. *Moreover,*
$r_{\otimes}(A) = \max\{t : t \in \sigma_{ap}(A)\}$.

Also $\{r_{e_j}(A) : j \in \mathbb{N}\} \subset \sigma_{ap}(A)$.

Spectral mapping theorem in max algebra for $A \in M_+^{\infty\times\infty}$:

True for σ_p and σ_{ap} **for polynomials without the absolute term**.

Theorem 11 *Let* $A \in M_+^{\infty\times\infty}$, $q \in \mathcal{P}_+$, $q = \sum_{j=1}^{\deg q} \alpha_j z^j$, $q \neq 0$. *Then*

$\sigma_p(q_\otimes(A)) = q_\otimes(\sigma_p(A))$ *and* $\sigma_{ap}(q_\otimes(A)) = q_\otimes(\sigma_{ap}(A))$.

In general we have:

Theorem 12 *Let $A \in M_{+}^{\infty\times\infty}$, $q \in \mathcal{P}_{+}$, $q = \sum_{j=0}^{\deg q} \alpha_j z^j$. Then*

$$q_{\otimes}(\sigma_p(A)) \subset \sigma_p(q_{\otimes}(A)) \subset q_{\otimes}(\sigma_p(A)) \cup \{\alpha_0\} \qquad (2)$$

and $$q_{\otimes}(\sigma_{ap}(A)) \subset \sigma_{ap}(q_{\otimes}(A)) \subset q_{\otimes}(\sigma_{ap}(A)) \cup \{\alpha_0\}.$$

- In general for σ_p nothing better than (2) can be said:

  Example: for **right shift** S^T we have $\sigma_p(S^T) = \emptyset$

  For the polynomial $q(z) = z + 1$ we have
  $\sigma_p(q_{\otimes}(S^T)) = \sigma_p(I \oplus S^T) = \{1\}$

- corresponding results are valid for more general **max** and **max-plus** type kernel operators and its **tropical** versions (**Bellman operators**)