The Erdos density theorem revisited

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partly joint work with Jorge Almeida

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• \mathcal{H} is a complex Hilbert space; $\mathcal{B}(\mathcal{H})$ is the set of bounded linear operators on \mathcal{H}

• projection P in B(H)

$$P^2 = P$$
 and $P^* = P$

• P, Q projections

$$P \leq Q$$
 if $PQ = P (= QP)$

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• The set of projections together with the partial order relation " \leq " is a complete lattice.

• Nest N - a totally ordered family of projections $N \subseteq B(H)$ containing 0 and the identity I

• Complete nest N - if N is a complete sublattice of the lattice of projections in $B(\mathcal{H})$

• Nest algebra $\mathcal{T}(\mathcal{N})$

all operators $T\in B(\mathcal{H})$ such that, for all $P\in\mathcal{N},$

 $T(P(\mathcal{H})) \subseteq P(\mathcal{H})$

equivalently

$$P^{\perp}TP = 0$$
 (with $P^{\perp} = I - P$)

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Examples

- Upper triangular $n \times n$ complex matrices
- Block upper triangular $n \times n$ complex matrices
- Volterra nest algebra $\mathcal{H} = L^2[0,1]$ (Lebesgue measure) P_t projection onto the space of functions f such that f = 0a.e. on [t,1]

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 $\mathcal{N} = \{ P_t : 0 \le t \le 1 \}$

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 $\mathcal{T}(\mathcal{N})$ is called the Volterra nest algebra

•
$$P \in \mathcal{N}$$

$$P_{-} = \bigvee \{ Q \in \mathcal{N} : Q < P \}$$

• Continuous nest ${\cal N}$

 $P_- = P$ for all $P \in \mathcal{N}$

• Continuous nest algebra $\mathcal{T}(\mathcal{N})$ – nest \mathcal{N} is continuous

The Volterra nest algebra is a continuous nest algebra. The nest algebras of the other examples are not continuous

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Finite rank operators are decomposable (Ringrose '65)

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Every finite rank operator in a nest algebra can be written as a finite sum of rank 1 operators lying in the nest algebra.

Finite rank operators are dense (J.A. Erdos '68)

Theorem

The set of finite rank operators lying in the unit ball of a nest algebra is dense in the ball for the strong operator topology.

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 \bullet Ideal ${\cal I}$ complex subspace ${\cal I}$ of the nest algebra ${\cal T}({\cal N})$ s. t.

 $\mathcal{T}(\mathcal{N})\mathcal{I}\subseteq\mathcal{I}$ and $\mathcal{I}\mathcal{T}(\mathcal{N})\subseteq\mathcal{I}$

• homomorphism $\varphi : \mathcal{N} \to \mathcal{N}$

$$\forall P, Q \in \mathcal{N} \qquad P \leq Q \implies \varphi(P) \leq \varphi(Q)$$

• left order continuous homomorphism If $\mathcal{M}\subseteq\mathcal{N},$ then

$$\varphi(\bigvee \mathcal{M}) = \bigvee \varphi(\mathcal{M})$$

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• Ideal \mathcal{I}

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${\mathcal I}$ weakly closed ideal

1° Finite rank operators in \mathcal{I} are decomposable (Erdos, Power '82) 2° (Erdos, Power '82)

Theorem

 $\mathcal{T}(\mathcal{N})$ nest algebra, \mathcal{I} weakly closed ideal. Then

$$\mathcal{I} = \{ T \in B(\mathcal{H}) : \tilde{P}^{\perp} T P = 0 \},\$$

where $P \mapsto \tilde{P}$ is a left order continuous homomorphism on the nest \mathcal{N} such that $\tilde{P} \leq P$ for each $P \in \mathcal{N}$.

 \tilde{P} is the projection onto $\overline{\text{span}\left(\bigcup_{T\in\mathcal{I}}TP(\mathcal{H})\right)}^{\|\cdot\|}$

(Proof uses Erdos' density theorem and decomposability (cf. 1°).)

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Nest algebra $\mathcal{T}(\mathcal{N})$ with product

$$T \circ S = \frac{1}{2}(TS + ST)$$

Complex subspace \mathcal{J} Jordan ideal

 $\mathcal{J} \circ \mathcal{T}(\mathcal{N}) \subseteq \mathcal{J}$

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Nest algebra $\mathcal{T}(\mathcal{N})$ with product

$$T \circ S = \frac{1}{2}(TS + ST) \qquad [T, S] = TS - ST$$

Complex subspace \mathcal{J} Jordan ideal

 $\mathcal{J} \circ \mathcal{T}(\mathcal{N}) \subseteq \mathcal{J}$

 $[\mathcal{L},\mathcal{T}(\mathcal{N})]\subseteq \mathcal{L}$

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• rank 1 operator $x \otimes y : \mathcal{H} \to \mathcal{H}$

$$z \mapsto \langle z, x \rangle y$$
 $x, y, z \in \mathcal{H}$

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• (Ringrose '65) $x \otimes y \in \mathcal{T}(\mathcal{N})$ iff $P_{-}x = 0$ and Py = y ($P \in \mathcal{N}$) where

$$P = \bigwedge \{ Q \in \mathcal{N} : Qy = y \}$$

 Consequence: If the nest N is continuous, ther

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• rank 1 operator $x \otimes y : \mathcal{H} \to \mathcal{H}$

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 $x \perp y$

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$$P_y = \bigwedge \{ Q \in \mathcal{N} : Qy = y \}$$

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$$P_y y = y$$
 and $\hat{P}_x x = 0$

• When the nest is continuous:

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$$x \otimes y \in \mathcal{T}(\mathcal{N})$$
 iff $P_y \leq \hat{P}_x$

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$$x \otimes y \in \mathcal{T}(\mathcal{N}) \quad \text{iff} \quad P_y \leq \hat{P}_x$$

 $x \otimes y \in \mathcal{T}(\mathcal{N}) \quad \Rightarrow \quad P_y z$

$$\Rightarrow P_y x = 0$$

Theorem

 $\mathcal{T}(\mathcal{N})$ nest algebra (respectively, continuous nest algebra) \mathcal{M} norm closed Jordan ideal (respectively, Lie ideal) $x \otimes y \in \mathcal{M}$ and $w \otimes z \in B(\mathcal{H})$ satisfying

$$\hat{P}_x \leq \hat{P}_w$$
 and $P_z \leq P_y$.

Then, $w \otimes z \in \mathcal{M}$.

The "corner" of
$$x \otimes y$$

$$\begin{bmatrix} 0 & P_y \mathcal{T}(\mathcal{N}) \hat{P}_x^{\perp} \\ 0 & 0 \end{bmatrix} = 0$$

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Proposition

 $\mathcal{T}(\mathcal{N})$ nest algebra (respectively, continuous nest algebra) \mathcal{M} norm closed Jordan ideal (respectively, Lie ideal), Let, for all $P \in \mathcal{N}$,

$$P' = \bigvee \left\{ P_y \in \mathcal{N} : x \otimes y \in \mathcal{M} \land \hat{P}_x < P \right\}$$
(1)

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Then

the mapping P → P' is a left order continuous homomorphism
 P' < P for all P ∈ N

Characterisation of the rank 1 operators in $\ensuremath{\mathcal{M}}$

Lemma

 $\mathcal{T}(\mathcal{N})$ nest algebra (respectively, continuous nest algebra) \mathcal{M} norm closed Jordan ideal (respectively, Lie ideal),

Then

 $x \otimes y \in \mathcal{M}$ if and only if, for all projections $P \in \mathcal{N}$,

$$P'^{\perp}(x\otimes y)P=0$$

Here $P \mapsto P'$ is the left order continuous homomorphism defined above.

Decomposability of the finite rank operators in $\ensuremath{\mathcal{L}}$

Theorem

 $\mathcal{T}(\mathcal{N})$ continuous nest algebra, \mathcal{L} norm closed Lie ideal $T \in \mathcal{L}$ finite rank operator

Then

T can be written as a finite sum of rank one operators lying in \mathcal{L} .

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III. Finite rank operators

 $P \mapsto P' = \bigvee \left\{ P_{\mathbf{y}} \in \mathcal{N} : \mathbf{x} \otimes \mathbf{y} \in \mathcal{J} \land \hat{P}_{\mathbf{x}} < P \right\}$

Characterisation of the finite rank operators in \mathcal{L}

Theorem

 $\mathcal{T}(\mathcal{N})$ continuous nest algebra, \mathcal{L} norm closed Lie ideal, T finite rank operator

Then

 $T \in \mathcal{L}$ if and only if, for all projections $P \in \mathcal{N}$,

 $P'^{\perp}TP=0$

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III. Finite rank operators

 $\boldsymbol{P} \mapsto \boldsymbol{P}' = \bigvee \left\{ \boldsymbol{P}_{\boldsymbol{y}} \in \mathcal{N} : \boldsymbol{x} \otimes \boldsymbol{y} \in \mathcal{J} \land \hat{\boldsymbol{P}}_{\boldsymbol{x}} < \boldsymbol{P} \right\}$

Proof. Consequence of the decomposability of the finite rank operators and the characterisation of rank 1 operators in \mathcal{L} .

recall the lemma $x\otimes y\in \mathcal{L}$ iff ${P'}^{\perp}(x\otimes y)P=0$

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Continuity of the nest is important

Let • \mathcal{N} - nest such that

$$\exists_{P\in\mathcal{N}} \quad \dim(P-P_{-})(\mathcal{H}) \geq 2.$$

• ${\mathcal L}$ - norm closed subspace generated by the projection $P-P_-$ and

$$\left\{S\in\mathcal{T}(\mathcal{N}):S=P_{-}SP_{-}^{\perp}+(P-P_{-})SP^{\perp}
ight\}$$
 (associative ideal)



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L is a norm closed Lie ideal and does not contain any (finite rank) operator T satisfying

$$T=(P-P_{-})T(P-P_{-}),$$

apart from those operators lying in the span of $P - P_{-}$.

e Hence none of the results presented for the finite rank operators apply to the norm closed Lie ideal *L*. Example

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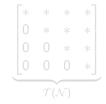
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Question: Does the decomposability of the finite rank operators still hold if the nest has only atoms of dimension one?

Answer: No.

Counter-example:

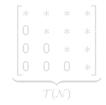


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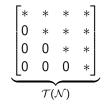


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Counter-example:



$$\mathcal{L} = \operatorname{span}\{I\}$$

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III. Jordan ideals

 $P \mapsto P' = \bigvee \left\{ P_{\mathbf{y}} \in \mathcal{N} : \mathbf{x} \otimes \mathbf{y} \in \mathcal{J} \land \hat{P}_{\mathbf{x}} < P \right\}$

(Lu-Yu; O.)

Theorem

 $\mathcal{T}(\mathcal{N})$ nest algebra, \mathcal{J} weakly closed subspace The following assertions are equivalent:

- $\textcircled{O} \ \mathcal{J} \ \textit{is a weakly closed Jordan ideal}$
- ② There exists a left order continuous homomorfism P → P' on N such that

$$\mathcal{J} = \{ T \in B(\mathcal{H}) : P'^{\perp} TP = 0 \text{ for all } P \in \mathcal{N} \}$$

 $\bigcirc \mathcal{J}$ is a weakly closed (associative) ideal

$\mathcal{T}(\mathcal{N})$ nest algebra

diagonal of $\mathcal{T}(\mathcal{N})$ $\mathcal{D}(\mathcal{N}) = \mathcal{T}(\mathcal{N}) \cap \mathcal{T}(\mathcal{N})^*$

(Hudson, Marcoux, Sourour '98)

Theorem

 \mathcal{L} weakly closed Lie ideal Then, there exist $\mathcal{K}(\mathcal{L})$ weakly closed associative ideal $\mathcal{D}_{\mathcal{K}(\mathcal{L})}$ von Neumann subalgebra of the diagonal such that

 $\mathcal{K}(\mathcal{L}) \subseteq \mathcal{L} \subseteq \mathcal{K}(\mathcal{L}) + \mathcal{D}_{\mathcal{K}(\mathcal{L})}$

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$$\mathcal{K}(\mathcal{L}) = \overline{\operatorname{span}\{PTP^{\perp} : T \in \mathcal{L}, P \in \mathcal{N}\}}^{w}$$

 $\mathcal{K}(\mathcal{L})$ is constructed starting with diagonal disjoint building blocks and, in some situations, $\mathcal{K}(\mathcal{L})$ is itself diagonal disjoint.

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$$\mathcal{D}_{\mathcal{K}(\mathcal{L})} = \{ D \in \mathcal{D}(\mathcal{N}) : \forall P \in \mathcal{N}_0 \; \exists \lambda \in \mathbb{C} \; D(P - \widetilde{P}) = \lambda(P - \widetilde{P}) \}$$

with $P \mapsto \widetilde{P}$ homomorphism associated with $\mathcal{K}(\mathcal{L})$ (Erdos and Power) and

$$\mathcal{N}_0 = \{ P \in \mathcal{N} \ : \ \widetilde{P} < P_- \}$$

$$\mathcal{K}(\mathcal{L}) = \overline{\mathsf{span}\{PTP^{\perp} : T \in \mathcal{L}, P \in \mathcal{N}\}}^{w}$$

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$$\mathcal{D}_{\mathcal{K}(\mathcal{L})} = \{ D \in \mathcal{D}(\mathcal{N}) \, : \, \forall P \in \mathcal{N}_0 \, \exists \lambda \in \mathbb{C} \, D(P - \widetilde{P}) = \lambda(P - \widetilde{P}) \}$$

with

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$$\mathcal{N}_0 = \{ P \in \mathcal{N} \ : \ \widetilde{P} < P_- \}$$

joint work with J. Almeida

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Definition

- \mathcal{L} Lie ideal $P \in \mathcal{N}$
 - condition I $(P P_{-})\mathcal{T}(\mathcal{N})(P P_{-}) \subseteq \mathcal{L}$ define

$$P'' = \bigvee \{ P_y : \exists_{x \in \mathcal{H}} x \otimes y \in \mathcal{L} \land \hat{P}_x < P \}$$

Condition II otherwise define

$P'' = \bigvee \{ P_y : \exists_{x \in \mathcal{H}} x \otimes y \in \mathcal{L} \land \hat{P}_x < P \land P_y \le P_- \}$

joint work with J. Almeida

Definition

- \mathcal{L} Lie ideal $P \in \mathcal{N}$
 - condition I $(P P_{-})\mathcal{T}(\mathcal{N})(P P_{-}) \subseteq \mathcal{L}$ define

$$P'' = \bigvee \{ P_y : \exists_{x \in \mathcal{H}} x \otimes y \in \mathcal{L} \land \hat{P}_x < P \}$$

Condition II otherwise define

$$\mathcal{P}'' = \bigvee \{ \mathcal{P}_y : \exists_{x \in \mathcal{H}} x \otimes y \in \mathcal{L} \land \hat{\mathcal{P}}_x < \mathcal{P} \land \mathcal{P}_y \leq \mathcal{P}_- \}$$

Proposition

 \mathcal{L} Lie ideal $P \mapsto P''$ left order continuous homomorphism

Definition

 ${\cal L}$ Lie ideal

$\mathcal{J}(\mathcal{L}) = \{ X \in B(\mathcal{H}) \, : \, (I - P'')XP = 0 \quad \text{for all } P \text{ in } \mathcal{N} \}$

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• FACT: $\mathcal{J}(\mathcal{L})$ associative ideal

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Theorem

 \mathcal{L} weakly closed Lie ideal Then $\mathcal{J}(\mathcal{L})$ is the largest weakly closed associative ideal contained in \mathcal{L}

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$\mathcal{K}(\mathcal{L}) \subseteq \mathcal{J}(\mathcal{L}) \subseteq \mathcal{L}$

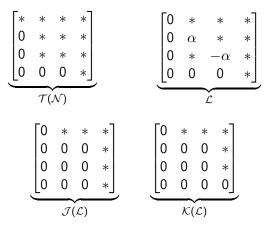
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$$\mathcal{K}(\mathcal{L}) \subseteq \mathcal{J}(\mathcal{L}) \subseteq \mathcal{L}$$

Example. $\mathcal{J}(\mathcal{L})$ might be strictly larger than $\mathcal{K}(\mathcal{L})$



$\ensuremath{\mathcal{L}}$ weakly closed Lie ideal

Definition

Define $D_{\mathcal{L}}$ as the projection onto the subspace $\bigvee_{P \in \mathcal{N}} (P - P'')(\mathcal{H})$.

Definition

$$\check{\mathcal{D}}(\mathcal{L}) = \{ D \in \mathcal{D}_{\mathcal{J}(\mathcal{L})} \, : \, D(I - D_{\mathcal{L}}) = 0 \}$$

Proposition

 $\breve{\mathcal{D}}(\mathcal{L})$ is

weakly closed *-subalgebra of $\mathcal{D}_{\mathcal{J}(\mathcal{L})}$ unital algebra, with unit $D_{\mathcal{L}}$ associative ideal of $\mathcal{D}_{\mathcal{J}(\mathcal{L})}$.

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Theorem

 ${\cal L}$ weakly closed Lie ideal Then

$$\mathcal{J}(\mathcal{L}) \subseteq \mathcal{L} \subseteq \mathcal{J}(\mathcal{L}) \oplus \breve{\mathcal{D}}(\mathcal{L})$$

$$\mathcal{K}(\mathcal{L})\subseteq\mathcal{J}(\mathcal{L})\subseteq\mathcal{L}\subseteq\mathcal{J}(\mathcal{L})\oplusreve{\mathcal{D}}(\mathcal{L})=\mathcal{K}(\mathcal{L})+\mathcal{D}_{\mathcal{K}(\mathcal{L})}$$
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