The Erdos density theorem revisited

Lina Oliveira

Instituto Superior Técnico
Universidade de Lisboa

partly joint work with Jorge Almeida

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I. Subspace lattices

• $\mathcal{H}$ is a complex Hilbert space; $B(\mathcal{H})$ is the set of bounded linear operators on $\mathcal{H}$

• projection P in $B(H)$

$$P^2 = P \quad \text{and} \quad P^* = P$$

• P, Q projections

$$P \leq Q \qquad \text{if} \qquad PQ = P \; (= QP)$$

• The set of projections together with the partial order relation "$\leq$" is a complete lattice.

I. Subspace lattices

- **Nest** $\mathcal{N}$ - a totally ordered family of projections $\mathcal{N} \subseteq B(\mathcal{H})$ containing 0 and the identity I

- **Complete nest** $\mathcal{N}$ - if $\mathcal{N}$ is a complete sublattice of the lattice of projections in $B(\mathcal{H})$

- **Nest algebra** $\mathcal{T}(\mathcal{N})$

all operators $T \in B(\mathcal{H})$ such that, for all $P \in \mathcal{N}$,

$$T(P(\mathcal{H})) \subseteq P(\mathcal{H})$$

equivalently $$P^{\perp}TP = 0 \qquad (\text{with} \quad P^{\perp} = I - P)$$

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Examples

- Upper triangular $n \times n$ complex matrices
- Block upper triangular $n \times n$ complex matrices
- Volterra nest algebra
  $\mathcal{H} = L^2[0,1]$ (Lebesgue measure)
  P_t projection onto the space of functions f such that $f = 0$ a.e. on $[t, 1]$

  $\mathcal{N} = \{P_t : 0 \leq t \leq 1\}$

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I. Subspace lattices

- $P \in \mathcal{N}$

$$P_- = \bigvee \{Q \in \mathcal{N} : Q < P\}$$

- **Continuous nest** $\mathcal{N}$

$$P_- = P \qquad \text{for all} \quad P \in \mathcal{N}$$

- **Continuous nest algebra** $\mathcal{T}(\mathcal{N})$ – nest $\mathcal{N}$ is continuous

The Volterra nest algebra is a continuous nest algebra.
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Finite rank operators are decomposable (Ringrose '65)

Lemma

Every finite rank operator in a nest algebra can be written as a finite sum of rank 1 operators lying in the nest algebra.

Finite rank operators are dense (J.A. Erdos '68)

Theorem

The set of finite rank operators lying in the unit ball of a nest algebra is dense in the ball for the strong operator topology.

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II. Ideals

- **Ideal $\mathcal{I}$**

complex subspace $\mathcal{I}$ of the nest algebra $\mathcal{T}(\mathcal{N})$ s. t.

$$\mathcal{T}(\mathcal{N})\mathcal{I} \subseteq \mathcal{I} \qquad \text{and} \qquad \mathcal{I}\mathcal{T}(\mathcal{N}) \subseteq \mathcal{I}$$

- **homomorphism $\varphi : \mathcal{N} \to \mathcal{N}$**

$$\forall P, Q \in \mathcal{N} \qquad P \leq Q \implies \varphi(P) \leq \varphi(Q)$$

- **left order continuous homomorphism**

If $\mathcal{M} \subseteq \mathcal{N}$, then

$$\varphi(\bigvee \mathcal{M}) = \bigvee \varphi(\mathcal{M})$$

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II. Ideals

$\mathcal{I}$ weakly closed ideal

1° Finite rank operators in $\mathcal{I}$ are decomposable (Erdos, Power '82)

2° (Erdos, Power '82)

Theorem

$\mathcal{T}(\mathcal{N})$ nest algebra, $\mathcal{I}$ weakly closed ideal. Then

$$\mathcal{I} = \{T \in B(\mathcal{H}) : \tilde{P}^{\perp} T P = 0\},$$

where $P \mapsto \tilde{P}$ is a left order continuous homomorphism on the nest $\mathcal{N}$ such that $\tilde{P} \leq P$ for each $P \in \mathcal{N}$.

$\tilde{P}$ is the projection onto $\overline{\operatorname{span}\left(\bigcup_{T\in\mathcal{I}} TP(\mathcal{H})\right)}^{\|\cdot\|}$

(Proof uses Erdos' density theorem and decomposability (cf. 1°).)

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Nest algebra $\mathcal{T}(\mathcal{N})$ with product

$$T \circ S = \frac{1}{2}(TS + ST)$$

Complex subspace $\mathcal{J}$

Jordan ideal

$$\mathcal{J} \circ \mathcal{T}(\mathcal{N}) \subseteq \mathcal{J}$$

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Nest algebra $\mathcal{T}(\mathcal{N})$ with product

$$T \circ S = \frac{1}{2}(TS + ST) \qquad [T, S] = TS - ST$$

Complex subspace $\mathcal{J}$

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$$\mathcal{J} \circ \mathcal{T}(\mathcal{N}) \subseteq \mathcal{J}$$

Complex subspace $\mathcal{L}$

Lie ideal

$$[\mathcal{L}, \mathcal{T}(\mathcal{N})] \subseteq \mathcal{L}$$

III. Finite rank operators

- **rank 1 operator** $\color{red}{x \otimes y} : \mathcal{H} \to \mathcal{H}$

$$z \mapsto \langle z, x \rangle y \qquad x, y, z \in \mathcal{H}$$

- (Ringrose '65)
  $x \otimes y \in \mathcal{T}(\mathcal{N})$ iff $P_- x = 0$ and $Py = y$ $(P \in \mathcal{N})$

  where

$$P = \bigwedge \{Q \in \mathcal{N} : Qy = y\}$$

- Consequence:
  If the nest $\mathcal{N}$ is continuous, then $x \perp y$

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$$P = \bigwedge \{Q \in \mathcal{N} : Qy = y\}$$

- Consequence:
  If the nest $\mathcal{N}$ is continuous, then $\quad x \perp y$

Projections associated to $x \otimes y$

- $$\hat{P}_x = \bigvee\{Q \in \mathcal{N} : Qx = 0\}$$
- $$P_y = \bigwedge\{Q \in \mathcal{N} : Qy = y\}$$
- $P_y y = y$ and $\hat{P}_x x = 0$
- When the nest is continuous:
  1. $x \otimes y \in \mathcal{T}(\mathcal{N})$ iff $P_y \leq \hat{P}_x$
  2. $x \otimes y \in \mathcal{T}(\mathcal{N}) \Rightarrow P_y x = 0$

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III. Finite rank operators

Theorem

$\mathcal{T}(\mathcal{N})$ nest algebra (respectively, continuous nest algebra)
$\mathcal{M}$ norm closed Jordan ideal (respectively, Lie ideal)
$x \otimes y \in \mathcal{M}$ *and* $w \otimes z \in B(\mathcal{H})$ *satisfying*

$$\hat{P}_x \leq \hat{P}_w \qquad \text{and} \qquad P_z \leq P_y .$$

Then, $w \otimes z \in \mathcal{M}$.

The "corner" of $x \otimes y$

$$\left[\begin{array}{c|c} 0 & P_y \mathcal{T}(\mathcal{N}) \hat{P}_x^{\perp} \\ \hline 0 & 0 \end{array}\right]$$

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III. Finite rank operators

Proposition

$\mathcal{T}(\mathcal{N})$ nest algebra (respectively, continuous nest algebra)
$\mathcal{M}$ norm closed Jordan ideal (respectively, Lie ideal),
Let, for all $P \in \mathcal{N}$,

$$P' = \bigvee \left\{ P_y \in \mathcal{N} : x \otimes y \in \mathcal{M} \;\wedge\; \hat{P}_x < P \right\} \tag{1}$$

Then

- *the mapping $P \mapsto P'$ is a left order continuous homomorphism*
- $$P' \leq P \qquad\qquad \text{for all} \quad P \in \mathcal{N}$$

Characterisation of the rank 1 operators in $\mathcal{M}$

Lemma

$\mathcal{T}(\mathcal{N})$ nest algebra (respectively, continuous nest algebra) $\mathcal{M}$ norm closed Jordan ideal (respectively, Lie ideal),

Then

$x \otimes y \in \mathcal{M}$ *if and only if, for all projections* $P \in \mathcal{N}$,

$$P'^{\perp}(x \otimes y)P = 0$$

Here $P \mapsto P'$ is the left order continuous homomorphism defined above.

III. Finite rank operators

Decomposability of the finite rank operators in $\mathcal{L}$

Theorem

$\mathcal{T}(\mathcal{N})$ continuous nest algebra, $\mathcal{L}$ norm closed Lie ideal
$T \in \mathcal{L}$ finite rank operator

Then

T can be written as a finite sum of rank one operators lying in $\mathcal{L}$.

Characterisation of the finite rank operators in $\mathcal{L}$

Theorem

$\mathcal{T}(\mathcal{N})$ continuous nest algebra, $\mathcal{L}$ norm closed Lie ideal, T finite rank operator

Then

$T \in \mathcal{L}$ *if and only if, for all projections* $P \in \mathcal{N}$,

$$P'^{\perp} T P = 0$$

III. Finite rank operators

$P \mapsto P' = \bigvee \left\{ P_y \in \mathcal{N} : x \otimes y \in \mathcal{J} \ \wedge \ \hat{P}_x < P \right\}$

Proof. Consequence of the decomposability of the finite rank operators and the characterisation of rank 1 operators in $\mathcal{L}$.

recall the lemma

$$x \otimes y \in \mathcal{L} \quad \text{iff} \quad P'^{\perp}(x \otimes y)P = 0$$

Continuity of the nest is important

Let

- $\mathcal{N}$ - nest such that

$$\exists_{P\in\mathcal{N}} \qquad \dim(P-P_{-})(\mathcal{H})\geq 2.$$

- $\mathcal{L}$ - norm closed subspace generated by the projection $P-P_{-}$ and

$$\left\{S\in\mathcal{T}(\mathcal{N}) : S = P_{-}SP_{-}^{\perp} + (P-P_{-})SP^{\perp}\right\} \quad (\text{associative ideal})$$

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$$\left\{ S \in \mathcal{T}(\mathcal{N}) : S = P_- S P_-^\perp + (P - P_-) S P^\perp \right\} \quad \big(\text{associative ideal}\big)$$

1. $\mathcal{L}$ is a norm closed Lie ideal and does not contain any (finite rank) operator T satisfying

$$T = (P - P_-)T(P - P_-),$$

apart from those operators lying in the span of $P - P_-$.

2. Hence none of the results presented for the finite rank operators apply to the norm closed Lie ideal $\mathcal{L}$.

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III. Finite rank operators

Question: Does the decomposability of the finite rank operators still hold if the nest has only atoms of dimension one?

Answer: No.

Counter-example:

$$\underbrace{\begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}}_{\mathcal{T}(\mathcal{N})} \qquad \mathcal{L} = \text{span}\{I\}$$

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III. Jordan ideals

$P \mapsto P' = \bigvee \left\{ P_y \in \mathcal{N} : x \otimes y \in \mathcal{J} \ \wedge \ \hat{P}_x < P \right\}$

(Lu–Yu; O.)

Theorem

$\mathcal{T}(\mathcal{N})$ nest algebra, $\mathcal{J}$ weakly closed subspace
The following assertions are equivalent:

1. *$\mathcal{J}$ is a weakly closed Jordan ideal*
2. *There exists a left order continuous homomorfism $P \mapsto P'$ on $\mathcal{N}$ such that*

$$\mathcal{J} = \{ T \in B(\mathcal{H}) : \quad P'^{\perp} T P = 0 \quad \text{for all } P \in \mathcal{N} \}$$

3. *$\mathcal{J}$ is a weakly closed (associative) ideal*

V. Lie ideals

$\mathcal{T}(\mathcal{N})$ nest algebra

diagonal of $\mathcal{T}(\mathcal{N})$ $\qquad \mathcal{D}(\mathcal{N}) = \mathcal{T}(\mathcal{N}) \cap \mathcal{T}(\mathcal{N})^*$

(Hudson, Marcoux, Sourour '98)

Theorem

$\mathcal{L}$ weakly closed Lie ideal
Then, there exist
$\mathcal{K}(\mathcal{L})$ weakly closed associative ideal
$\mathcal{D}_{\mathcal{K}(\mathcal{L})}$ von Neumann subalgebra of the diagonal
such that

$$\mathcal{K}(\mathcal{L}) \subseteq \mathcal{L} \subseteq \mathcal{K}(\mathcal{L}) + \mathcal{D}_{\mathcal{K}(\mathcal{L})}$$

V. Lie ideals

$$\mathcal{K}(\mathcal{L}) = \overline{\text{span}\{PTP^{\perp} \,:\, T \in \mathcal{L}, \quad P \in \mathcal{N}\}}^{w}$$

$\mathcal{K}(\mathcal{L})$ is constructed starting with diagonal disjoint building blocks and, in some situations, $\mathcal{K}(\mathcal{L})$ is itself diagonal disjoint.

$$\mathcal{D}_{\mathcal{K}(\mathcal{L})} = \{D \in \mathcal{D}(\mathcal{N}) \,:\, \forall P \in \mathcal{N}_0 \; \exists \lambda \in \mathbb{C} \; D(P - \widetilde{P}) = \lambda(P - \widetilde{P})\}$$

with

$P \mapsto \widetilde{P}$ homomorphism associated with $\mathcal{K}(\mathcal{L})$ (Erdos and Power)

and

$$\mathcal{N}_0 = \{P \in \mathcal{N} \,:\, \widetilde{P} < P_-\}$$

V. Lie ideals

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and

$$\mathcal{N}_0 = \{P \in \mathcal{N} \ : \ \widetilde{P} < P_-\}$$

Definition

$\mathcal{L}$ Lie ideal $\quad P \in \mathcal{N}$

1. condition I $\quad (P - P_-)\mathcal{T}(\mathcal{N})(P - P_-) \subseteq \mathcal{L}$
   define

$$P'' = \bigvee \{P_y \; : \; \exists_{x \in \mathcal{H}} \; x \otimes y \in \mathcal{L} \wedge \hat{P}_x < P\}$$

2. condition II $\quad$ otherwise
   define

$$P'' = \bigvee \{P_y \; : \; \exists_{x \in \mathcal{H}} \; x \otimes y \in \mathcal{L} \wedge \hat{P}_x < P \wedge P_y \leq P_-\}$$

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V. Lie ideals

Proposition

$\mathcal{L}$ Lie ideal $\quad P \mapsto P''\quad$ left order continuous homomorphism

Definition

$\mathcal{L}$ Lie ideal

$$\mathcal{J}(\mathcal{L}) = \{X \in B(\mathcal{H}) : (I - P'')XP = 0 \quad \text{for all } P \text{ in } \mathcal{N}\}$$

- FACT: $\mathcal{J}(\mathcal{L})$ associative ideal

V. Lie ideals

Proposition

$\mathcal{L}$ *Lie ideal* $\quad P \mapsto P''$ *left order continuous homomorphism*

Definition

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V. Lie ideals

Theorem

$\mathcal{L}$ weakly closed Lie ideal
Then
$\mathcal{J}(\mathcal{L})$ is the largest weakly closed associative ideal contained in $\mathcal{L}$

$$\mathcal{K}(\mathcal{L}) \subseteq \mathcal{J}(\mathcal{L}) \subseteq \mathcal{L}$$

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V. Lie ideals

Example. $\mathcal{J}(\mathcal{L})$ might be strictly larger than $\mathcal{K}(\mathcal{L})$

$$\underbrace{\begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{bmatrix}}_{\mathcal{T}(\mathcal{N})} \qquad \underbrace{\begin{bmatrix} 0 & * & * & * \\ 0 & \alpha & * & * \\ 0 & * & -\alpha & * \\ 0 & 0 & 0 & * \end{bmatrix}}_{\mathcal{L}}$$

$$\underbrace{\begin{bmatrix} 0 & * & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \end{bmatrix}}_{\mathcal{J}(\mathcal{L})} \qquad \underbrace{\begin{bmatrix} 0 & * & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathcal{K}(\mathcal{L})}$$

V. Lie ideals

$\mathcal{L}$ weakly closed Lie ideal

Definition

Define $D_{\mathcal{L}}$ as the projection onto the subspace $\bigvee_{P \in \mathcal{N}} (P - P'')(\mathcal{H})$.

Definition

$\breve{\mathcal{D}}(\mathcal{L}) = \{D \in \mathcal{D}_{\mathcal{J}(\mathcal{L})} : D(I - D_{\mathcal{L}}) = 0\}$

Proposition

$\breve{\mathcal{D}}(\mathcal{L})$ is

weakly closed *-subalgebra of $\mathcal{D}_{\mathcal{J}(\mathcal{L})}$

unital algebra, with unit $D_{\mathcal{L}}$

associative ideal of $\mathcal{D}_{\mathcal{J}(\mathcal{L})}$.

V. Lie ideals

$\mathcal{L}$ weakly closed Lie ideal

Definition

Define $D_{\mathcal{L}}$ as the projection onto the subspace $\bigvee_{P \in \mathcal{N}}(P - P'')(\mathcal{H})$.

Definition

$\check{\mathcal{D}}(\mathcal{L}) = \{D \in \mathcal{D}_{\mathcal{J}(\mathcal{L})} \, : \, D(I - D_{\mathcal{L}}) = 0\}$

Proposition

$\check{\mathcal{D}}(\mathcal{L})$ is

weakly closed *-subalgebra of $\mathcal{D}_{\mathcal{J}(\mathcal{L})}$

unital algebra, with unit $D_{\mathcal{L}}$

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V. Lie ideals

$\mathcal{L}$ weakly closed Lie ideal

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Define $D_{\mathcal{L}}$ as the projection onto the subspace $\bigvee_{P \in \mathcal{N}} (P - P'')(\mathcal{H})$.

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V. Lie ideals

Theorem

$\mathcal{L}$ weakly closed Lie ideal
Then

$$\mathcal{J}(\mathcal{L}) \subseteq \mathcal{L} \subseteq \mathcal{J}(\mathcal{L}) \oplus \check{\mathcal{D}}(\mathcal{L})$$

$$\mathcal{K}(\mathcal{L}) \subseteq \mathcal{J}(\mathcal{L}) \subseteq \mathcal{L} \subseteq \mathcal{J}(\mathcal{L}) \oplus \check{\mathcal{D}}(\mathcal{L}) = \mathcal{K}(\mathcal{L}) + \mathcal{D}_{\mathcal{K}(\mathcal{L})} .$$