The total graphs of finite commutative (semi)rings

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LAW'14, Ljubljana

S semigroup/semiring/nearring/poset/... $\longrightarrow$ $\Gamma(S)$

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Beck ['88], Akbari, Anderson, Badawi, DeMeyer, Livingston, Mohammadian, Mulay, Redmond, . . . ['99-]

Anderson, Badawi ['08]

S semiring/ring $\longrightarrow$ $\tau(S)$

$Z(S)\ldots$ zero-divisors of S

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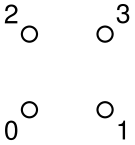
$\tau(S)$ is the total graph of S, if:

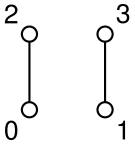
- vertices: all elements $x \in S$
- $x - y$ is an edge $\iff$ $x \neq y$ and $x + y$ is a zero-divisor.

Examples



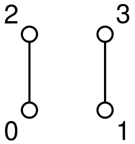
$\tau(\mathbb{Z}_4)$

Examples



$\tau(\mathbb{Z}_4) = 2K_2$

Examples



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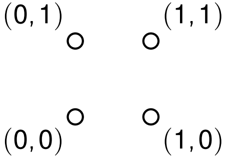

$\tau(\mathbb{Z}_2 \times \mathbb{Z}_2)$

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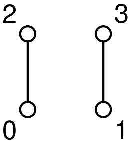

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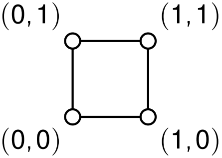

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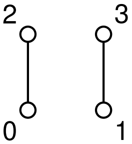

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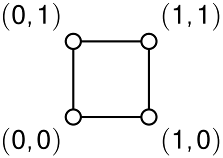

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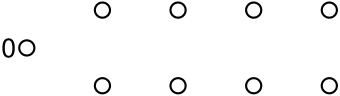

$\tau(F)$

F a field, $char(F) \neq 2$

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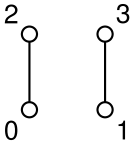

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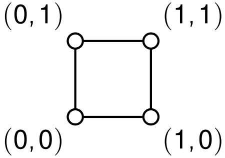

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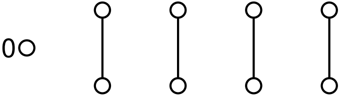

$\tau(F) = K_1 \cup K_2 \cup K_2 \cup \dots$

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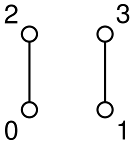

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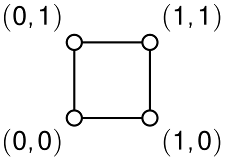

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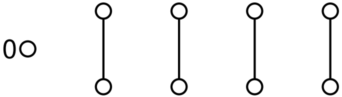

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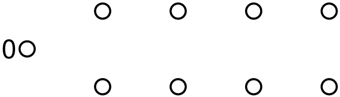

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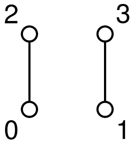

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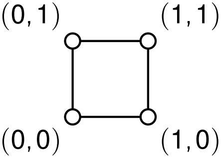

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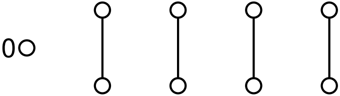

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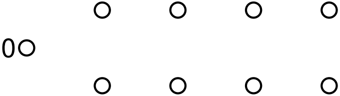

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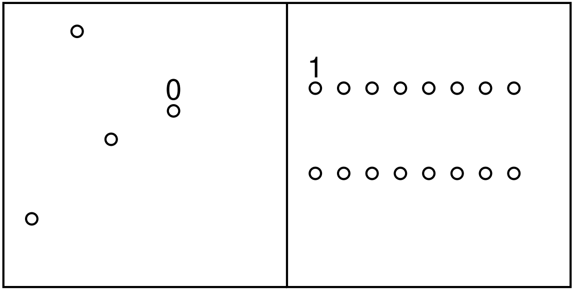
$Z(R) \triangleleft R$ $\qquad$ $R - Z(R)$



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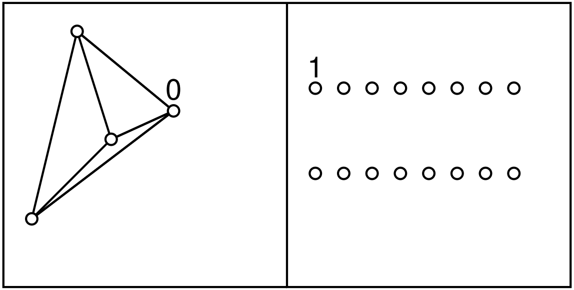

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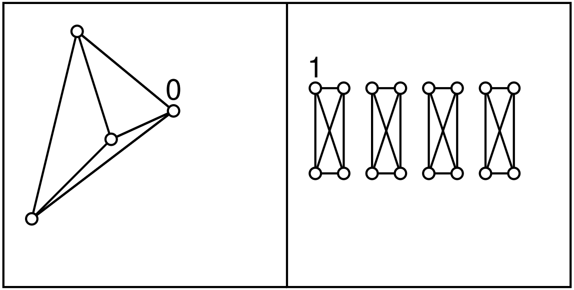

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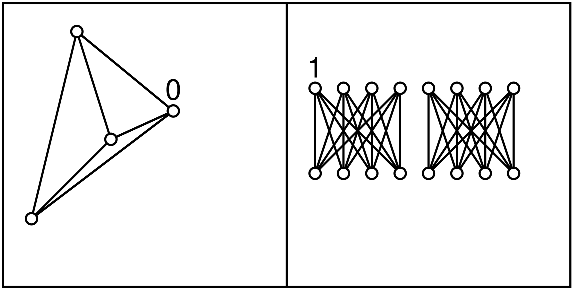

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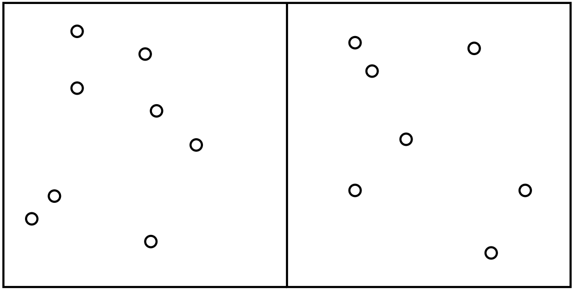

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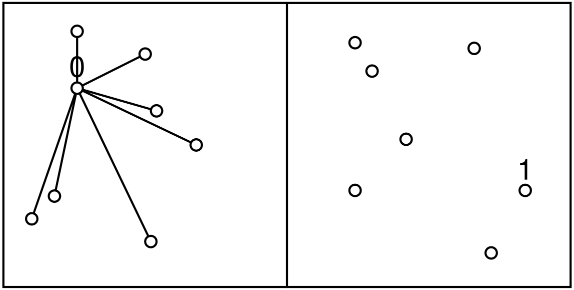
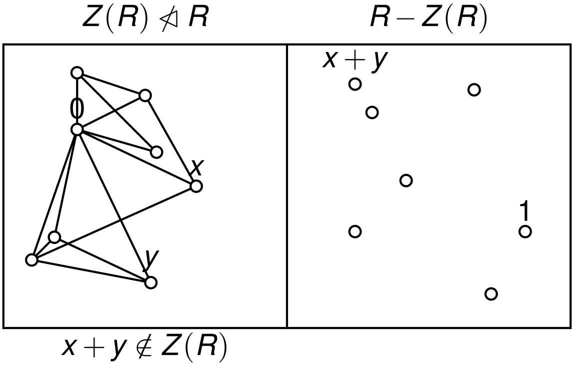
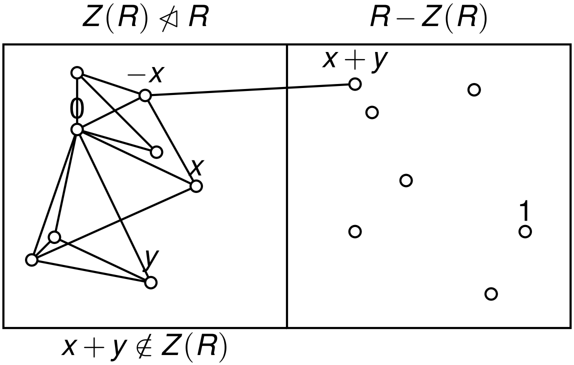




$Z(R) \ntriangleleft R$
$R - Z(R)$
0
x
y
$x + y$
1
$x + y \notin Z(R)$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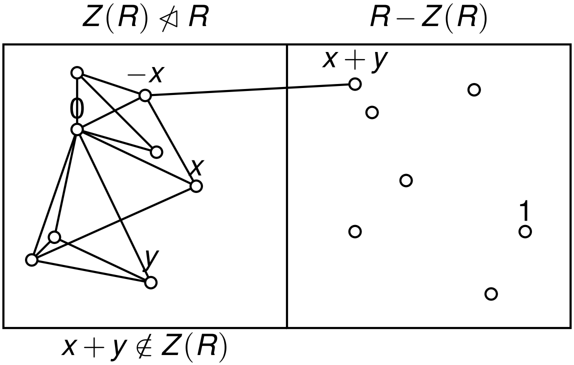

Theorem (Anderson, Badawi, '08)

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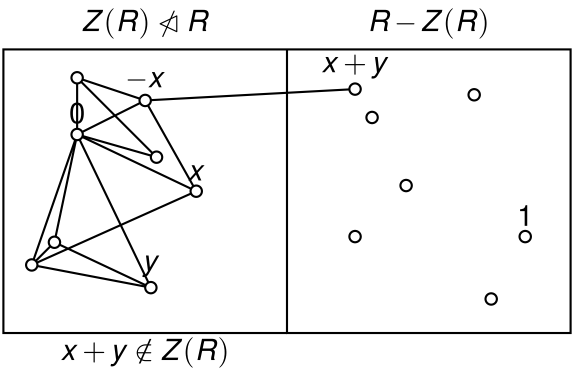
If $\tau(R)$ connected, then $\mathrm{diam}(\tau(R)) = d(0,1)$.



Theorem (Anderson, Badawi, ’08)

R commutative ring.
If $\tau(R)$ *connected, then* $\mathrm{diam}(\tau(R)) = d(0,1)$*.*
If R finite, then $\mathrm{diam}(\tau(R)) = 2$*.*

Theorem

If F is a field and $n \geq 2$, then

$$\operatorname{diam}(\tau(M_n(F))) = 2$$

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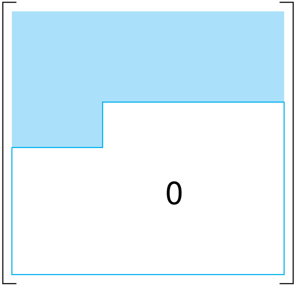

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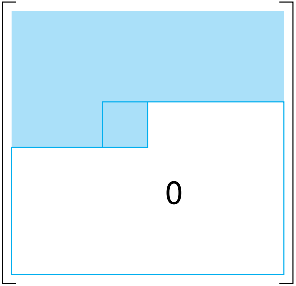

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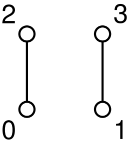
Theorem

$|R| < \infty$

$\tau(R)$ is Hamiltonian if and only if R is not local.

Examples



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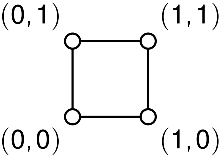

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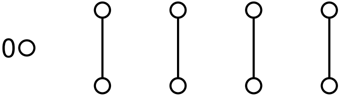

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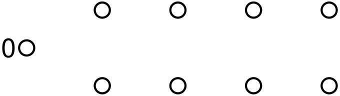

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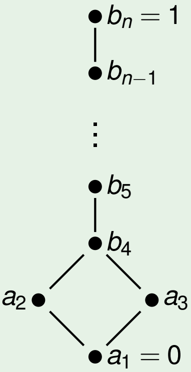
$DL_n =$



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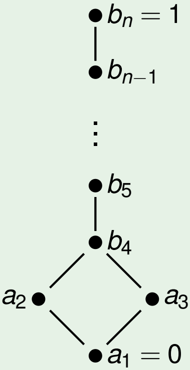

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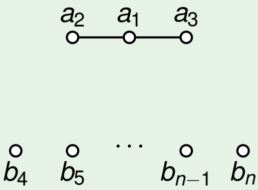

Example

$SP_4 = \{0, 1, 2, b\}$

$+$	0	1	2	b
0	0	1	2	b
1	1	2	1	2
2	2	1	2	1
b	b	2	1	0

$\cdot$	0	1	2	b
0	0	0	0	0
1	0	1	2	b
2	0	2	2	0
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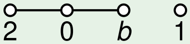

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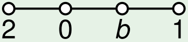

Theorem

If S a finite commutative semiring and $4 \leq \mathrm{girth}(\tau(S)) < \infty$, then $S \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

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$\frac{\lvert S\rvert - 1}{2} K_2 \cup K_1$	S a field, $char(S) \neq 2$
$K_2 \cup mK_1$	
$2K_2$	$R \cong \mathbb{Z}_4$
$P_3 \cup nK_1$, $n \geq 1$	
P_4	

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If S is a finite commutative semiring and $\mathrm{girth}(\tau(S)) = \infty$, then

$\tau(S)$	S
$\lvert S \rvert K_1$	S a field, $char(S) = 2$
	S is an antinegative entire semiring
$\frac{\lvert S \rvert - 1}{2} K_2 \cup K_1$	S a field, $char(S) \neq 2$
$K_2 \cup mK_1$	
$2K_2$	$R \cong \mathbb{Z}_4$
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$\tau(S)$	S
$\lvert S \rvert K_1$	S a field, $char(S) = 2$
	S is an antinegative entire semiring
$\frac{\lvert S \rvert - 1}{2} K_2 \cup K_1$	S a field, $char(S) \neq 2$
$K_2 \cup mK_1$	
$2K_2$	$R \cong \mathbb{Z}_4$
$P_3 \cup nK_1,\ n \geq 1$	S contains a subsemiring $\cong DL_4$
P_4	

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$\frac{\lvert S\rvert-1}{2} K_2 \cup K_1$	S a field, $char(S) \neq 2$
$K_2 \cup mK_1$	$S = T \cup \{a\}$, T antinegative entire semiring, $a^2 = 0$ and $ta = a$ for all $t \in T - \{0\}$
$2K_2$	$R \cong \mathbb{Z}_4$
$P_3 \cup nK_1$, $n \geq 1$	S contains a subsemiring $\cong DL_4$
P_4	$S \cong SP_4$