Semitransitivity

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Introduction

- A collection $\mathcal{L} \subseteq \mathcal{M}_n(\mathbb{C})$ of complex $n \times n$ matrices is said to be transitive, or that it acts transitively on $V = \mathbb{C}^n$, if

$$\forall x, y \in V \setminus 0 \; \exists A \in \mathcal{L} \ni Ax = y.$$

  Or, equivalently,

$$\forall x \in \mathcal{V} \setminus 0, \mathcal{L}x = \{Ax : A \in \mathcal{L}\} \supseteq V \setminus 0.$$

- We say that $\mathcal{L}$ is semitransitive, it that it acts semitransitively, if

$$\forall x, y \in \mathcal{V} \setminus 0 \; \exists A \in \mathcal{L} \ni Ax = y \text{ or } Ay = x.$$

  Or, equivalently,

$$\forall x, y \in V \setminus 0, \; x \in \mathcal{L}y \text{ or } y \in \mathcal{L}x.$$

- Topological version(s) are defined in the obvious way.
- Introduced by H. Rosenthal and V. Troitsky in early 2000's.

Observations

- If $\mathcal{L}$ is a group, then semitransitivity= transitivity.
- If $\mathcal{L}$ is transitive then $\mathcal{L}$ is irreducible: the only invariant subspaces are 0 and V.
- If $\mathcal{L}$ is semitransitive, then the set of invariant subspaces is totally ordered: if U, W are invariant subspaces, then $U \leq W$ or $W \leq U$.
- If $\mathcal{L}$ is a unital semigroup , then $\mathcal{L}$ is semitransitive if and only if the set of orbits is totally ordered.
- If $\mathcal{L}$ is a unital algebra, then $\mathcal{L}$ is semitransitive if and only if the set of invariant subspaces is totally ordered.
- The set of all upper triangular Toeplitz matrices is semitransitive.
- Compressions of semitransitive collections are semitransitive, i.e., if $E \in \mathcal{M}_n(\mathbb{C})$ is an idempotent, and $\mathcal{L}$ is semitransitive, then so is $E\mathcal{L}E \subseteq L(EU, EU)$.

Semigroups

Joint work with Bernik, Grunenfelder, Radjavi, Troitsky:

- Case $n = 1$ is nontrivial:
  - semigroup $\mathcal{L} \subseteq \mathbb{C}$ is semitransitive if and only if it contains $T = \{\text{roots of } 1\}$ and the preimage of the positive cone of some total order on $\mathbb{C}^\times / T$
  - the only compact subsemigroup of $\mathbb{C}$ is the unit disc
- If $n > 1$, then semitransitive semigroups do not necessarily contain minimal semitransitive subsemigroups.
- If $\mathcal{L}$ is semitransitive and the rank of every nonzero matrix in $\mathcal{L}$ is k, then k divides n.
- For every k that divides n there exists a constant-rank-k semitransitive semigroup.

Constant-rank-k semitransitive semigroup

The semigroup consisting of all matrices of the block form (all blocks are $k \times k$)

$$\begin{pmatrix} 0 & \cdots & 0 & A_1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & A_2 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & A_i & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where $i = 1, \ldots, n/k$, block-entries $A_1, \ldots, A_{i-1}$ are arbitrary and A_i invertible (these entries are in the i-th column).

Associative Algebras

Burnside:

- An algebra $\mathcal{A}$ of matrices is transitive if and only if $\mathcal{A} = \mathcal{M}_n(\mathbb{C})$.

BGMRT:

- An algebra of matrices is semitransitive if and only if, up to simultaneous similarity, it contains all upper triangular Toeplitz matrices.
- Idea of the proof: We prove that $\mathcal{A}$ contains a nilpotent matrix of maximal rank.
  - Block-triangularize $\mathcal{A}$ in a suitable way.
  - Prove that the projections onto the superdiagonal blocks must be nonzero - and are hence full.
  - Use linear combinations to produce an upper-triangular nilpotent element with nonzero super-diagonal entries.

Linear spaces

Duality:

- Let $\mathcal{L}$ be a subspace of $L(U,V)$. Then we can view U as a subspace of $L(\mathcal{L},V)$ via

$$\widehat{(\ )}\colon U \to L(\mathcal{L},V), \quad u \mapsto \widehat{u}$$

  given by

$$\widehat{u}A = Au.$$

Properties of $\mathcal{L}$ can be translated into properties of $\widehat{U}$.

- In particular: $\mathcal{L}$ is transitive if and only if every nonzero $\widehat{x} \in \widehat{U}$ is surjective.

Questions:

- Semitransitivity only makes sense if $U = V$. Can you generalize the notion of semitransitivity to the case where $U \neq V$ (in a natural and interesting way)?

- Can you describe semitransitivity of $\mathcal{L}$ in terms of intrinsic properties of $\widehat{U}$ (i.e., without invoking the identification $U \equiv \widehat{U}$)?

Spatial (or flag-type?) semitransitivity

Partial answer:

- Let
$$0 = V_0 < V_1 < \ldots < \; V_k = V$$
be the lattice of all invariant subspaces of $\mathcal{L} \subseteq L(V, V)$. If for every $i = 1, \ldots, k$ and
$$\forall x \in V_i \setminus V_{i-1}, \; \mathcal{L}x = V_i,$$
then $\mathcal{L}$ is semitransitive.
- This condition is equivalent to saying that the range of every $\widehat{x} \in \widehat{V}_i \setminus \widehat{V}_{i-1}$ is V_i, i.e., for every $x \in V_i \setminus V_{i-1}$ we have that $\widehat{x}$ viewed as a map from $\mathcal{L}$ to V_i is surjective.

There exist irreducible spaces that are semitransitive, but not transitive; so in general spatial semitransitivity is stronger then semitransitivity. However, if $k = n = \dim V$, i.e., $\mathcal{L}$ is triangularizable, then spatial semitransitivity is equivalent to semitransitivity.

Question(s):

- What can we say about structure and properties of spatially semitransitive spaces (possibly with additional structure and properties)? In particular:
  - Do spatially semitransitive spaces contain triangularizable semitransitive spaces?
  - When are spatial semitransitivity and transitivity the same?
- How about for nonlinear collections of matrices (perhaps with a topological notion of spatial semitransitivity)?

Some answers in the context of algebraic groups will be discussed later.

Some properties of semitransitive spaces

Radjavi, Trotsky:

- If $\mathcal{L} \subseteq \mathcal{M}_n(\mathbb{C})$ is semitransitive, then there exists a cyclic vector for $\mathcal{L}$.
- Hence the dimension of a semitransitive space is at least n.

Working group at LAW Bled: the above holds over an arbitrary field.

Contrast that with transitivity: the minimal possible dimension of a transitive spaces over $\mathbb{C}$ is $2n+1$, but can be as small as n over other fields. For example

$$\left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{R} \right\} \subseteq \mathcal{M}_2(\mathbb{R})$$

is transitive.

If $n = 2$, then it is easy to see that there are, up to simultaneous similarity only 3 semitransitive spaces. (Exercise!)

For larger n there does not seem to be any hope of classifying semitransitive spaces. For $n > 5$ even the structure of the ones of dimension n seems completely inaccessible.

In particular:

- If $n > 2$, then there exist irreducible minimal semitransitive spaces that are not transitive.
- If $n > 2$, then there exists minimal semitransitive spaces of dimension $> n$ that are triangular.
- If $n \geq 4$ then semitransitive spaces of dimension n do not need to contain rank-one matrices.

Semitransitive spaces of minimal dimension

Bernik, Drnovšek, Kokol-Bukovšek, Košir, Omladič:

- Semitransitive spaces of dimension n are triangularizable.

A few words about the proof:

- Use the following observation: If $\mathcal{L} \subseteq L(V, W)$ is transitive, then $\dim \mathcal{L} \geq \dim V + \dim W + 1$.
- Study rank varieties of $\widehat{V}$: $X_i = \{v \in V : \dim(\mathcal{L}v) \leq i\}$.
- Prove that these varieties are spaces and hence they must coincide with the lattice of invariant subspaces.

More questions

- Are there minimal semitransitive spaces that are transitive?
- Given that $\mathcal{L} + \mathcal{L}^*$ is transitive, find necessary and sufficient conditions for $\mathcal{L}$ to be semitransitive.
- The curious case of infinite dimensions?

Semitransitive Jordan algebras

Bernik, Drnovšek, Kokol-Bukovšek, Košir, Omladič:

- Spatial semitransitivity is equivalent to semitransitivity, in particular, irreductible semitransitive Jordan algebras are transitive.
- Up to simultaneous similarity, every semitransitive Jordan algebra contains all upper triangular Toeplitz matrices.
- In particular, every minimal semitransitive Jordan algebra is similar to the algebra of upper triangular Toeplitz matrices and is therefore triangularizable and of dimension n.

Prehomogeneous vector spaces and semitransitivity

This is joint work with J. Bernik; some slides have been recycled from a J. B. presentation.

- Assume that $V = \mathbb{C}^n$ and that $G \subseteq GL_n(\mathbb{C}) = GL(V)$ is a linear algebraic group.
- G, or more precisely the action of G on V may have the following properties.
  1. V is a flag-type $\mathbb{C}G$ module: the poset $\{\overline{Gx} : x \in V\}$ of orbit closures coincides with a flag $0 = V_0 < V_1 < \ldots < V_k = V$ of linear subspaces of V.
  2. G is (Zariski) semitransitive: for each pair $x, y \in V \setminus 0$ we have that either $x \in \overline{Gy}$ or $y \in \overline{Gx}$.
  3. there are finitely many orbits for the action of G on V.
  4. (G, V) is a prehomogeneous vector space: there exists a Zariski-dense orbit.
- We have (1) $\implies$ (2) $\implies$ (3) $\implies$ (4), all implications are strict.

Prehomoheneous vector spaces

- Prehomogeneous vector spaces have been studied for over 50 years now, especially in connection with Fourier analysis, (semi)-invariants, zeta functions, arithmetic,...

- Observation: if $v \in V$ is such that Gv is Zariski open in V, then $\dim(Gv) = \dim(V)$. On the other hand, since $g \mapsto gv$ is a surjective morphism, we have $\dim(G) \geq \dim(Gv) = n$.

- A major achievement was obtained by Sato and Kimura in 1977 who classified irreducible prehomogeneous vector spaces up to castling transformations (a method of obtaining a higher dimensional V' over a higher dimensional group G' starting from (G, V).) They found 32 cases, some containing infinite families, of reduced irreducible prehomoheneous vector spaces.

- If V is an irreducible (as a $\mathbb{C}G$-module), then G is reductive, so their results deal with reductive linear algebraic groups. Later research in the area is also mainly concentrated on reductive groups.

- For a reductive G we can not have V of flag-type unless the flag is trivial.

Finitely Many Orbits

- If action of G on V has only finitely many orbits, then (G, V) is prehomogoneous vector space, but not all prehomogeneous vector spaces have finitely many orbits. Irreducible ones that do have finitely many orbits were classified by Kac in 1980.
- Classification over algebraically closed fiels of prime characteristic was obtained by Guralnick, Liebeck, Macpherson and Seitz in 1997. The lists essentially overlap, there are some additional cases in positive characteristic and some cases missing in small characteristics.
- $G = GL_2 \otimes GL_2 \subseteq GL_4$ is an example with finitely many orbits that is not of flag-type.

Flag-type $\mathbb{C}G$-modules and semitransitive Lie algebras

- The study of flag-type $\mathbb{C}G$-modules can be translated into Lie algebras.
- Let $\mathfrak{g}$ be the Lie algebra of G. Then V is a flag-type $\mathbb{C}G$-module if and only if $\mathfrak{g} \subseteq \mathcal{M}_n$ is spatially semitransitive, i.e., there exists a flag $0 = V_0 < V_1 < \ldots < V_k = V$ such that for every $v \in V_i \setminus V_{i-1}$ we have $\mathfrak{g}v = V_i$.
- It turns out that a Lie algebra is spatially semitransitive if and only if it is semitransitive
- Main ingredient: If $\mathfrak{g}$ is semitransitive and irreducible, and $n > 1$, then $\mathfrak{s} = [\mathfrak{g}, \mathfrak{g}]$ is either $\mathfrak{sl}_n$(with n arbitrary) or $\mathfrak{sp}_n$ (with n even). In particular this means that $\mathfrak{s}$ is transitive.

This follows by adjusting an argument in 1970 paper *Infinite dimensional primitive Lie algebras* by V. Guillemin (who in turn credits the it to S. Sternberg):

- Note that $\mathfrak{s} = [\mathfrak{g}, \mathfrak{g}]$ is simple and $\mathfrak{g} \subseteq \mathfrak{s} + \mathbb{C}\mathrm{I}$.
- Use semitransitivity of $\mathfrak{s} + \mathbb{C}\mathrm{I}$ to show that $\alpha = \nu^+ - \nu^-$, where ν^+ and ν^- are maximal and minimal weights for the action, is a root.
- From classification of simple Lie algebras it then follows that this can only happen if $\mathfrak{s}$ is of type A or C and V may be identified with its natural module or its dual.

- Analysing in great detail the action of the radical $\mathfrak{r}$ of $\mathfrak{g}$ on V we prove the following consequence: Every Borel subalgebra of a semitransitive Lie algebra is also semitransitive. In particular, every minimal semitransitive Lie algebra is solvable (i.e., triangularizable).

- The most delicate part of this analysis is suitably 'excising' parts of V where $\mathfrak{s}$ acts ad-trivially on the radical $\mathfrak{r}$ of $\mathfrak{g}$ - in linear algebraic terms these are parts of V corresponding to 1×1 blocks in the block-triangularization of $\mathfrak{g}$.

Semitransitive Lie algebras of minimal dimension

- They are triangularizable.
- They do contain a nilpotent of maximal rank. $(= n - 1)$
- The space of nilpotents is of dimension $n - 1$.
- Do not have to contain nilpotents of rank 1.
- Not unique.

- If the diagonal is not constant (i.e., they are not nilpotent as Lie algebras), then they contain a diagonal matrix of the form

$$\mathrm{diag}(a, a+1, \ldots, a+(n-1))$$

  and hence they are $\mathbb{N}$-graded.
  - The main tool is dimension counting together with analysing the adjoint action of $\mathfrak{g}$ on itself, more precisely the action of the semi-simple part of ad_s on $\mathfrak{g}$, where $s \in \mathfrak{g}$ is a matrix whose (n, n)-entry is nonzero.
- They are algebraic if and only if $a \in \mathbb{Q}$.
- Question: Which nilpotent Lie algebras can be represented as the space of nilpotent elements of some semitransitive Lie algebra of minimal dimension? Recently answered by J. Bernik and K. Šivic for graded filiform Lie algebras.

Structure of Flag-type $\mathbb{C}G$-modules

The Lie algebra work then translates into the following theorem:
The following statements are equivalent for an algebraic group $G \subseteq GL(V)$:

1. V is a flag-type $\mathbb{C}G$-module.
2. V^* is a flag-type $\mathbb{C}G$-module.
3. V is a flag-type $\mathbb{C}B$-module for any Borel subgroup B in G.
4. There exists a closed solvable subgroup $D \subseteq G$ such that V is a flag-type $\mathbb{C}D$-module.

Note that for solvable groups we have that flag-type = semitransitive. So we have that

V is a flag-type $\mathbb{C}G$ module if and only if G contains a triangularizable semitransitive subgroup.

More Questions

- How about positive characteristic?

- How about replacing the Zariski topology with some other topology? (the example of SU_2 gives a partial negative answer in case of Euclidean topology for compact Lie groups)

- How about semigroups?

Thank You!