On the length of matrix algebras

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Preliminaries

Let $\mathbb{F}$ be a field, let $\mathcal{A}$ be a finite-dimensional associative unitary $\mathbb{F}$-algebra and let $\mathcal{S} = \{a_1, \ldots, a_k\}$ be a finite generating set for $\mathcal{A}$.

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$$\mathcal{L}_0(\mathcal{S}) \subseteq \mathcal{L}_1(\mathcal{S}) \subseteq \cdots \subseteq \mathcal{L}_m(\mathcal{S}) = \mathcal{L}_{m+1}(\mathcal{S}) = \cdots = \mathcal{L}(\mathcal{S}) = \mathcal{A}.$$

Main Definitions

Definition

A number $l(\mathcal{S})$ is called *length of a finite generating set* $\mathcal{S}$ provided $l(\mathcal{S}) = \min\{m \in \mathbb{Z}_+ : \ \mathcal{L}_m(\mathcal{S}) = \mathcal{A}\}$.

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Since $\dim \mathcal{L}_i(\mathcal{S}) < \dim \mathcal{L}_{i+1}(\mathcal{S})$ unless $\mathcal{L}_i(\mathcal{S}) = \mathcal{A}$, then the trivial upper bound for the length is $l(\mathcal{A}) \leq \dim \mathcal{A} - 1$.

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Theorem (C.J. Pappacena, 1997)

Let $\mathbb{F}$ be an arbitrary field and let $\mathcal{A}$ be a finite-dimensional $\mathbb{F}$-algebra. Write $d = \dim \mathcal{A}$, $m = \max\{\deg a : a \in \mathcal{A}\}$. Then

$$l(\mathcal{A}) < m\sqrt{\frac{2d}{m-1} + \frac{1}{4}} + \frac{m}{2} - 2.$$

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Theorem (A. J.M. Spencer, R. S. Rivlin, 1959)

Let $\mathcal{S}$ be a set of K 3×3 matrices. Then
$l(\mathcal{S}) \leq 2$ if $K = 1$;
$l(\mathcal{S}) \leq 5$ if $K = 2$;
$l(\mathcal{S}) \leq K + 2$ if $K \geq 3$.

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Theorem (W.E. Longstaff, A.C. Niemeyer, Oreste Panaia, 2006) (M.S. Lambrou, W. E. Longstaff, 2009)

Let $n = 5, 6$, $\mathcal{S} = \{A, B\} \subset M_n(\mathbb{C})$, $\mathcal{L}(\mathcal{S}) = M_n(\mathbb{C})$. Then $l(\mathcal{S}) \leq 2n - 2$.

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Let $\mathbb{F}$ be an arbitrary field and let $\mathcal{A} \subseteq T_n(\mathbb{F})$. Then
1. $l(T_n(\mathbb{F})) = n - 1$;
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Upper bound:

$$\prod_{k=1}^{n} (A_k - (A_k)_{k,k} I) = 0 \text{ for any } A_1, \ldots, A_n \in T_n(\mathbb{F}).$$

Generating set for $T_n(\mathbb{F})$:

$$\mathcal{S} = \{E_{k,k}, E_{k,k+1}, E_{n,n} | k = 1, \ldots, n-1\}, \;\; l(\mathcal{S}) = n - 1.$$

Upper bound for the length of commutative matrix algebras

Theorem (A. Paz, 1984)

Let $\mathcal{A}$ be a commutative subalgebra in the full matrix algebra $M_n(\mathbb{C})$ over the field of complex numbers $\mathbb{C}$. Then $l(\mathcal{A}) \leq n-1$.

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Theorem (A.E. Guterman, O.V. Markova, 2009)

Let $\mathbb{F}$ be an arbitrary field and let $\mathcal{A}$ be a a commutative subalgebra in the matrix algebra $M_n(\mathbb{F})$. Then

1. $l(\mathcal{A}) \leq n-1$;
2. $l(\mathcal{A}) = n-1$ iff the algebra $\mathcal{A}$ is generated by a *nonderogatory* matrix C, i.e. by such a matrix $C \in M_n(\mathbb{F})$, that

$$\dim_{\mathbb{F}}(\langle C^0 = E_n,\ C,\ C^2, \ldots,\ C^{n-1}\rangle) = n.$$

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In the present talk we would like to provide a description of commutative subalgebras in $M_n(\mathbb{F})$ of the length $n-2$ (i.e. with length closest to maximal value) over algebraically closed fields.

The following theorem shows that the problem can be reduced to the case of nilpotent subalgebras of the length $n-2$:

Theorem

Let $\mathbb{F}$ be an algebraically closed field and let $n \in \mathbb{N}$, $n \geq 2$. Consider a commutative subalgebra $\mathcal{A}$ in $M_n(\mathbb{F})$ of the length $l(\mathcal{A}) = n-2$. Then there exist a number $m \in \mathbb{N}$, $2 \leq m \leq n$, a commutative subalgebra $\mathcal{B} \subseteq M_m(\mathbb{F})$ of the length $m-2$ and of the type $\mathbb{F}I + \mathcal{N}$, where $\mathcal{N}$ is a nilpotent algebra, and if $m < n$, a commutative subalgebra $\mathcal{C} \subseteq M_{n-m}(\mathbb{F})$ generated by a nonderogatory matrix such that $\mathcal{A}$ is conjugated to the algebra $\mathcal{B} \oplus \mathcal{C}$.

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Proof ideas: if $\mathcal{A}$ is not local itself, it is conjugated to a direct sum of some smaller commutative matrix algebras $\mathcal{R}$ and $\mathcal{T}$ of sizes r and t, $r+t=n$. For direct sum $l(\mathcal{R} \oplus \mathcal{T}) \leq l(\mathcal{R}) + l(\mathcal{T}) + 1$ (O.M., 2005). Hence $r+t-2 \leq r-r'+t-t'+1$, for some integers $r', t' \geq 1$, i.e. $r'=t'=1$ or wlog $r'=2, t'=1$. If the field is large, $l(\mathcal{R}) = r-1$ & $l(\mathcal{T}) = t-1 \Rightarrow l(\mathcal{A}) = n-1$. Thus, only the second option is possible.

Lemma

Let $n \in \mathbb{N}$, $n \geq 3$ and let $\mathbb{F}$ be an arbitrary field. Consider a commutative local subalgebra $\mathcal{A}$ in $M_n(\mathbb{F})$ of the type $\mathcal{A} = \mathbb{F}E + J(\mathcal{A})$. If $l(\mathcal{A}) = n - 2$, then the nilpotency index of the Jacobson radical of the algebra $\mathcal{A}$ is equal to $n - 1$.

Lemma

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Theorem (D.A. Suprunenko, R.I. Tyshkevich, 1966; I.A. Pavlov, 1967)

Let $n \geq 3$ and let $\mathbb{F}$ be an arbitrary field. Consider a commutative nilpotent subalgebra $\mathcal{B}$ in $M_n(\mathbb{F})$ with the nilpotency index equal to $n-1$, containing a matrix B which minimal polynomial has the degree $n-1$. Set $\mathcal{A} = \mathbb{F}E_n + \mathcal{B}$. Then the algebra $\mathcal{A}$ is conjugated in $M_n(\mathbb{F})$ with one of the following subalgebras:

$\mathcal{A}_{0;n} = \langle I_n, A, A^2, \ldots, A^{n-2}\rangle$, where $A = E_{1,2} + \cdots + E_{n-2,n-1} \in M_n(\mathbb{F})$;

$\mathcal{A}_{1;n} = \langle E_{1,n},\ C |\ C \in \mathcal{A}_{0;n}\rangle$;

$\mathcal{A}_{2;n} = \langle E_{n,n-1},\ C |\ C \in \mathcal{A}_{0;n}\rangle$;

$\mathcal{A}_{3;n}(\alpha) = \langle E_{1,n} + \alpha E_{n,n-1},\ C |\ C \in \mathcal{A}_{0;n}\rangle$, $n \geq 4, \alpha \in \mathbb{F},\ \alpha \neq 0$.

Description of commutative matrix algebras of the length $n-2$

Theorem

Let $n \geq 2$ and let $\mathbb{F}$ be an algebraically closed field. Consider a commutative subalgebra $\mathcal{A}$ in $M_n(\mathbb{F})$, containing the identity matrix I_n. Then $l(\mathcal{A}) = n-2$ iff the algebra $\mathcal{A}$ is conjugated in $M_n(\mathbb{F})$ with one of the following algebras:

1. $\mathbb{F}I_n$, if $n = 2$;

2. $\mathbb{F}I_2 \oplus \mathcal{C}_{n-2}$, where $\mathcal{C}_{n-2} \in M_{n-2}(\mathbb{F})$ is a subalgebra generated by a nonderogatory matrix;

3. $\mathcal{A}_{0;n}$;

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4. $\mathcal{A}_{1;n}$;

5. $\mathcal{A}_{2;n}$;

6. if $n = 4$, $\mathcal{A}_{3;4}(1)$;

7. if $n = 4$, $\mathrm{char}\mathbb{F} = 2$, $\mathcal{A}_{4;4} = \langle I_4, E_{1,2} + E_{3,4},\ E_{1,3} + E_{2,4},\ E_{1,4}\rangle$;

8. $\mathcal{A}_{j;m} \oplus \mathcal{C}_{n-m}$, where $j = 0, 1, 2$, $3 \leq m < n$, $\mathcal{C}_{n-m} \in M_{n-m}(\mathbb{F})$ is a subalgebra generated by a nonderogatory matrix.

Algebras of the types 3–7 are pairwise non-conjugate.

Exceptions

Theorem

Let $m, n \in \mathbb{N}$, $n \geq 4$ and let $\mathbb{F}$ be an arbitrary field and let $\mathcal{C} \subseteq M_m(\mathbb{F})$ be a subalgebra generated by a nonderogatory matrix.
Then for any $\alpha \in \mathbb{F}$, $\alpha \neq 0$ it holds that
1. $l(\mathcal{A}_{3;n}(\alpha)) = n - 3$ if $n \geq 5$;
2. $l(\mathcal{A}_{3;4}(\alpha) \oplus \mathcal{C}) \leq m + 1$.

Pappacena C. J. An upper bound for the length of a finite-dimensional algebra// Journal of Algebra. 1997. №. 197. P. 535-545.

Paz A. An Application of the Cayley–Hamilton theorem to matrix polynomials in several variables// Linear and Multilinear Algebra. 1984. V. 15. P. 161-170.

Guterman A. E., Markova O. V. Commutative matrix subalgebras and length function// Linear Algebra and its Applications. 2009. V. 430. P. 1790-1805.

Markova O.V., Characterization of commutative matrix subalgebras of maximal length over an arbitrary field// Vestn. Mosk. Univ., Ser. 1, **5**(2009), 53–55; English transl. in Mosc. Univ. Math. Bull., **64**:5(2009), 214–215.

Suprunenko D.A., Tyshkevich R.I., Commutative matrices, Academic Press, New York, NY, 1968; 2-nd edition, URSS, Moscow, 2003.

Pavlov I.A. On commutative nilpotent algebras of matrices// Dokl. Akad. Nauk BSSR **11**:10 (1967). P. 870–872.

THANK YOU!