On selfadjoint extensions of semigroups of partial isometries

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This project is joint work with

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Definition. *A semigroup* $(S, \circ)$ *is a non-empty set equipped with an associative binary operator* $\circ$.

If $x, y \in S$, then y is called an inverse of x if $yxy = y$ and $xyx = x$. A semigroup S in which each element has a unique inverse is called an inverse semigroup.

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Example. *Let X be a non-empty set. The symmetric inverse semigroup of X is:*

$$\mathcal{I}_X = \{f : A \to B : A \subseteq X, B \subseteq X, \text{ and } f \text{ is a bijection}\}.$$

For $f \in \mathcal{I}_X$, the (unique) inverse of $f : A \to B$ is

$$g : B \to A, \quad g(f(a)) = a.$$

The Wagner-Preston Theorem says that every inverse semigroup admits a representation as a subsemigroup of some symmetric inverse semigroup.

It was observed by Barnes (see also Duncan and Paterson) that this lifts to a faithful *-representation of $\mathcal{S}$ on $\ell_2(\mathcal{S})$ as a selfadjoint semigroup of **partial isometries**,

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Let $\mathcal{H}$ be a Hilbert space. A partial isometry is a bounded linear operator V on $\mathcal{H}$ satisfying one (and hence all) of the following:

(a) $V|_{(\ker V)^\perp} : (\ker V)^\perp \to \mathrm{ran}\, V$ is an isometry.

(b) $P_V := V^*V$ is a projection (called the initial projection of V).

(c) $Q_V := VV^*$ is a projection (called the final projection of V).

(d) $V = V V^* V$.

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If $\mathcal{S}$ is a selfadjoint semigroup of partial isometries, then the unique inverse of an element V of $\mathcal{S}$ is V^*.

A number of C^*-algebras are defined as the closed linear span of a selfadjoint semigroup of partial isometries:

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Theorem. *[Popov-Radjavi ∘ 2013]* *Let $\mathcal{S} = \mathcal{S}^*$ be an irreducible semigroup of partial isometries. Then we can write $\mathcal{H} = L^2(\Omega, \mathcal{K})$, where $\mathcal{K}$ is a Hilbert space and (Ω, μ) is a probability space, so that for each $V \in \mathcal{S}$ we have:*

- *there exist measurable sets X and Y in Ω so that $P_V\mathcal{H} = L^2(X, \mathcal{K})$ and $Q_V\mathcal{H} = L^2(Y, \mathcal{K})$;*
- *$V = U \int_X^{\oplus} V_t dt$, where $V_t \in \mathcal{B}(\mathcal{K})$ is a* **unitary** *operator for almost all $t \in X$, and $U : L^2(X, \mathcal{K}) \to L^2(Y, \mathcal{K})$ is a surjective isometry defined via:*

$$(Uf)(t) = w(t) f(\varphi^{-1}(t))$$

for some measure-preserving bijection $\varphi : X \to Y$ (modulo sets of measure zero) and a weight function $w : Y \to \mathbb{R}^+$.

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Let $k \in \mathbb{N} \cup \{\infty\}$, and let $\mathcal{U}$ be a semigroup of unitary operators acting on a Hilbert space $\mathcal{K}$.

We write $\mathcal{S}_0^k(\mathcal{U})$ to denote the semigroup of all $k \times k$ matrices having at most one non-zero entry which must then belong to $\mathcal{U}$.

Let $\mathcal{S}_1^k(\mathcal{U})$ denote the semigroup of all $k \times k$ matrices having at most one non-zero entry **in each row and each column**, and each such entry must belong to $\mathcal{U}$.

Theorem. *[Popov-Radjavi ∘ 2013] Suppose that $\mathcal{S}$ is an irreducible, norm-closed semigroup of partial isometries containing a non-zero compact operator. Then there exists $k \in \mathbb{N} \cup \{\infty\}$ and an irreducible group $\mathcal{U}$ of unitary matrices such that up to unitary equivalence,*

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Corollary. *[Popov-Radjavi ◦ 2013] Let $\mathcal{S}$ be an irreducible semigroup of partial isometries containing a non-zero compact operator. Then the selfadjoint semigroup $\mathcal{T}$ generated by $\mathcal{S}$ consists of partial isometries.*

Questions.

- Is the presence of the non-zero compact operator necessary?
- Is irreducibility a necessary condition?
- In general, which conditions guarantee that a semigroup of partial isometries on $\mathcal{H}$ can be extended to a selfadjoint semigroup of partial isometries?

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Question 1. Is every semigroup of partial isometries *-extendible?

For example, let $\mathcal{J}$ denote the semigroup of all isometries in $\mathcal{B}(\mathcal{H})$.

Theorem. *[Halmos-Wallen ∘ 1969]:*
Suppose that V and W are partial isometries. Then VW is a partial isometry if and only if $P_V = V^*V$ commutes with $Q_V = VV^*$.

Corollary. *In any *-extendible semigroup of partial isometries, $\mathcal{P}(\mathcal{S}) = \{P_V : V \in \mathcal{S}\}$ and $\mathcal{Q}(\mathcal{S}) = \{Q_V : V \in \mathcal{S}\}$ form two commuting families of projections, and $\mathcal{P}(\mathcal{S}) \subseteq \mathcal{Q}(\mathcal{S})'$.*

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The Halmos-Wallen Theorem provides us with a necessary condition for a semigroup $\mathcal{S}$ of partial isometries to be *-extendible, namely: $\mathcal{P}(\mathcal{S}) \cup \mathcal{Q}(\mathcal{S})$ must be a commuting family of projections.

Question 2. Suppose $\mathcal{S}$ is a semigroup of partial isometries for which $\mathcal{P}(\mathcal{S}) \cup \mathcal{Q}(\mathcal{S})$ is commutative. Is $\mathcal{S}$ *-extendible? •

The issue is to try to add adjoints of elements of $\mathcal{S}$ to $\mathcal{S}$. Suppose $A, B \in \mathcal{S}$ - is A^*B a partial isometry?

$$(A^*B)(A^*B)^* = A^*BB^*A = A^*Q_BA.$$

This is clearly selfadjoint. Also,

$$(A^*Q_BA)^2 = A^*Q_BAA^*Q_BA = A^*Q_BQ_AQ_BA = A^*Q_BA,$$

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What about products of three elements? Is A^*BC^* a partial isometry?

Example. *Let* $E = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ *and* $F = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. *Let*

$$A = \begin{bmatrix} 0 & E & 0 & 0 \\ 0 & 0 & 0 & 0 \\ & & 0 & 0 \\ & & 0 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 & 0 & 0 & I_2 \\ 0 & 0 & 0 & 0 \\ & & 0 & 0 \\ & & 0 & 0 \end{bmatrix}, \ C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ & & 0 & F \\ & & 0 & 0 \end{bmatrix}.$$

Set $\mathcal{S} = \{A, B, C, 0_8, I_8\}$. *Then* $\mathcal{S}$ *is a semigroup of partial isometries. Moreover,*

$$\mathcal{P}(\mathcal{S}) = \{0_8, I_8, \Delta(0, E, 0, 0), \Delta(0, 0, 0, I), \Delta(0, 0, 0, F)\},$$
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However,

$$A^*BC^* = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & EF & 0 \\ & & 0 & 0 \\ & & 0 & 0 \end{bmatrix}$$

is not a partial isometry.

Note: $\mathcal{S}$ is not irreducible. •

Question 3. Suppose that $\mathcal{S}$ is an irreducible semigroup of partial isometries, and that $\mathcal{P}(\mathcal{S}) \cup \mathcal{Q}(\mathcal{S})$ is commutative. Is $\mathcal{S}$ *-extendible?

This is substantially harder than the first two questions.

However,

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Now set $\mathcal{H}_0 = L^2[0,1]$;

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Here $|\varphi(t)| = |\psi(t)| = \sqrt{2}$ for all $t \in [0,1]$. (Need a special choice that doesn't concern us here.)
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There do exist positive results:

Theorem. *Suppose that $\mathcal{S}$ is an irreducible semigroup of partial isometries, that $\mathcal{P}(\mathcal{S}) \cup \mathcal{Q}(\mathcal{S})$ admits a minimal element. Then $\mathcal{S}$ is *-extendible.* •

Proposition. *Let $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$ be a semigroup of partial isometries for which $\mathcal{Q}(\mathcal{S})$ is commutative. Then there exists a semigroup $\mathcal{S}_{max}$ of partial isometries which is maximal with respect to the conditions that*

(a) $\mathcal{S}_{max} \supseteq \mathcal{S}$, *and*

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The proof of this is based on the following:

If $\mathcal{S}$ is a (WLOG - unital) semigroup of partial isometries and $\mathcal{Q}(\mathcal{S})$ is commutative, then given $T \in \mathcal{S}$, the semigroup $\mathcal{S}_1 := \langle \mathcal{S} \cup \{Q_T\} \rangle$ consists of partial isometries, and if $W = S_m Q_T S_{m-1} Q_T \cdots Q_T S_1$, then

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If $\mathcal{S}$ is a (WLOG - unital) semigroup of partial isometries and $\mathcal{Q}(\mathcal{S})$ is commutative, then given $T \in \mathcal{S}$, the semigroup $\mathcal{S}_1 := \langle \mathcal{S} \cup \{Q_T\} \rangle$ consists of partial isometries, and if $W = S_m Q_T S_{m-1} Q_T \cdots Q_T S_1$, then

$$\begin{aligned} Q_W &= Q_{S_m T} Q_{S_m S_{m-1} T} \cdots Q_{S_m S_{m-1} \cdots S_2 T} Q_{S_m S_{m-1} \cdots S_1} \\ &\in W^*(\mathcal{Q}(\mathcal{S})). \end{aligned}$$

The conclusion is that $W^*(\mathcal{Q}(\mathcal{S}_1)) = W^*(\mathcal{Q}(\mathcal{S}))$.

In fact, it can be shown that the conclusion holds if $E \in W^*(\mathcal{Q}(\mathcal{S}))$ is a projection and $\mathcal{S}_1 := \langle \mathcal{S} \cup \{E\} \rangle$.

Corollary. *Let $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$ be a semigroup of partial isometries and suppose that the von Neumann algebra $W^*(\mathcal{Q}(\mathcal{S}))$ generated by $\mathcal{Q}(\mathcal{S})$ forms a masa in $\mathcal{B}(\mathcal{H})$. Then the semigroup $\mathcal{T}$ generated by $\mathcal{S}$ and $\mathcal{S}^*$ consists of partial isometries.*

Note: For the semigroup $\mathcal{S}$ adapted from Read's example, $W^*(\mathcal{Q}(\mathcal{S}))$ had uniform infinite multiplicity.

Theorem. *Suppose $\mathcal{S}$ is a semigroup of partial isometries.*

- *Suppose that $\mathcal{P}(\mathcal{S}) = \mathcal{Q}(\mathcal{S}) \subseteq \mathcal{S}$. Then $\mathcal{S}$ is *-extendible.*
- *Hence, if $\mathcal{P}(\mathcal{S}) \cup \mathcal{Q}(\mathcal{S}) \subseteq \mathcal{S}$, then $\mathcal{S}$ is *-extendible.*

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Corollary. *Let $\mathcal{S}$ be a semigroup of partial isometries such that $W^*(\mathcal{Q}(\mathcal{S}))$ has uniform finite multiplicity. Then $\mathcal{S} \subseteq \mathcal{S}_0$, where $\mathcal{S}_0$ is a semigroup of partial isometries for which $\mathcal{P}(\mathcal{S}_0) \cup \mathcal{Q}(\mathcal{S}_0) \subseteq \mathcal{S}_0$.*

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Hence $\mathcal{S}_0$ is *-extendible, and thus $\mathcal{S}$ is *-extendible.

Conclusion: If $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$ is an irreducible semigroup of partial isometries and $\mathcal{Q}(\mathcal{S})$ is commutative, then $W^*(\mathcal{Q}(\mathcal{S}))$ has uniform multiplicity.

- If that multiplicity is finite, then $\mathcal{S}$ is *-extendible.
- The "trans-Read" example shows that if the multiplicity is infinite, then $\mathcal{S}$ need not be *-extendible.
- If $\mathcal{E} := \{E_{i,j} : 1 \leq i, j\} \subseteq \mathcal{B}(\ell_2)$ are the standard matrix units and $\mathcal{U}(\ell_2)$ denotes the group of all unitaries acting on ℓ_2, then $\mathcal{S} := \mathcal{E} \otimes \mathcal{U}(\ell_2)$ is an irreducible semigroup of partial isometries, $W^*(\mathcal{Q}(\mathcal{S}))$ has uniform infinite multiplicity, and $\mathcal{S} = \mathcal{S}^*$.

They laughed when I said I was going to be a comedian. Well, they're not laughing now.

Bob Monkhouse

Thank you for your attention.