Simultaneous Versions of Wieland's Positivity Theorem

(Joint Work with Gordon MacDonald and Alexey Popov)

Today "positive" or "nonnegative" operator shall mean an operator on a function space that takes nonnegative functions to themselves.

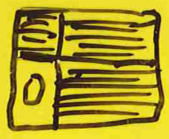
We write $T \gtrsim 0$ if T is a nonnegative operator.†

$$T \gtrsim S \quad \text{means} \quad T - S \gtrsim 0$$

† unless I forget and write $T \geq 0$.

In finite dimensions we fix a basis and consider matrices of operators. $M \in \mathcal{M}_n(\mathbb{C})$ is nonnegative if all its entries are nonnegative. If $M = (\lambda_{ij})$, then $|M| = (|\lambda_{ij}|)$ by definition.

<u>Wielandt</u> (1950):

Let M be an indecomposable matrix. (There is no permution matrix P such that $P^{-1}MP$ has the form .)

If $\rho(M) = \rho(|M|)$, then M is "effectively positivisable" that is, $\exists$ diagonal unitary D and $\lambda \in \mathbb{T}$ such that $\lambda D^{-1} M D = |M|$.

We'll consider operators on $\mathcal{X} = \mathcal{L}^2(X, \mu)$.

(almost everything is okay for $\mathcal{L}^p$ with $1 < p < \infty$.)

X Hausdorff-Lindelof

μ σ-finite regular Borel measure on X.

Def. A standard subspace ($\equiv$ lattice ideal) of $\mathcal{X}$ is any subspace of the form

$\mathcal{X}_0 = \{ f \in \mathcal{X} : f = 0 \text{ a.e. outside } X_0 \}$,

where X_0 is a borel subset of X.

We can identify $\mathcal{X}_0$ with $\mathcal{L}^2(X_0, \mu|X_0)$.

A set of operators, nonnegative or not, is called decomposable iff every member of it leaves a fixed, nontrivial, standard subspace invariant.

It follows from Perron-Frobenius and DePagter results that

if $K \gtrsim 0$ is an indecomposable compact operator on $\mathcal{X} = \mathcal{L}^2(X, \mu)$, then there is a unique idempotent $E \gtrsim 0$ of finite rank r such that

$$E = \lim_m \left(\frac{K}{\rho(K)}\right)^{rm},$$

where $\rho(K)$ is the spectral radius of K.

We call E <u>the Perron-Frobenius idempotent of</u> K. If $r > 1$, then relative to a standard decomposition of $\mathcal{X} = \mathcal{L}^p(X_1, \mu|X_1) + \cdots + \mathcal{L}^p(X_r, \mu|X_r)$

$$K = \begin{bmatrix} 0 & 0 & 0 & \cdots & K_r \\ K_1 & 0 & 0 & \cdots & 0 \\ 0 & K_2 & 0 & \cdots & 0 \\ \vdots & 0 & K_3 & \cdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & \vdots & \vdots & K_{r-1} & 0 \end{bmatrix} \text{ and } E = \begin{bmatrix} E_1 & 0 & \cdots & 0 \\ 0 & E_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & E_r \end{bmatrix}$$

Wielandt (1950): Let M be an indecomposable matrix.

(You can't permute the basis to get a form $\begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$ for M.)

If $\rho(M) = \rho(|M|)$, then M is "effectively positivisable", i.e., there is a diagonal (unitary) D and $\lambda \in \mathbb{T}$ such that $\lambda D^{-1} M D = |M|$.

Bernik-Marcoux-R (2012): If K is a regular operator on $\mathcal{L}^p(X,\mu)$ with $|K|$ power-compact and $\rho(K) = \rho(|K|)$, then there is $f \in \mathcal{L}^\infty(X,\mu)$ with $|f| = 1$ a.e. and $\lambda \in \mathbb{T}$ such that

$$\lambda M_f^{-1} K M_f = |K|.$$

$|T|$?

$T \in \mathcal{B}(\mathcal{L}^p(X,\mu))$ is <u>real</u> if it takes real-valued functions to themselves.

A real T is called <u>regular</u> if

$$T = R - S, \quad T \geq 0, \quad S \geq 0.$$

General T can be expressed as $T_1 + i T_2$, with T_1 and T_2 real. T is called <u>regular</u> if T_1 and T_2 are both regular.

Then the <u>modulus</u> of T can be defined as the nonnegative operator $|T|$ given by

$$|T| f = \sup_{|g| \leq f} |Tg|$$

for $f \geq 0$.

$\mathcal{S}$ will be a multiplicative semigroup of regular operators on $\mathcal{L}^p(X,\mu)$.

We'll also assume it is positive-homogeneous: $\alpha S \in \mathcal{S}$ for $S \in \mathcal{S}$ and $\alpha \in \mathbb{R}^+$.

<u>Theorem 1</u>. If

(1) $|S|$ is compact for all $S \in \mathcal{S}$,

(2) every nonzero $S \in \mathcal{S}$ is indecomposable,

(3) $\rho(S) = \rho(|S|)$ for all $S \in \mathcal{S}$, and

(4) $\overline{\mathcal{S}}$ contains no zero divisors,

then there is $\varphi \in \mathcal{L}^\infty(X,\mu)$, $|\varphi(x)| = 1$ a.e. such that for all $S \in \mathcal{S}$,

$$\lambda_S M_\varphi^{-1} S M_\varphi = |S|.$$

($\lambda_S \in \mathbb{T}$ may of course depend on S.)

Translation into f. d.

<u>Theorem</u>. If $\mathcal{S}$ is a positive-homogeneous semigroup $\subseteq \mathcal{M}_n(\mathbb{C})$ such that

(i) every nonzero $S \in \mathcal{S}$ is indecomposable,

(ii) $\rho(S) = \rho(|S|)$ for all $S \in \mathcal{S}$, and

(iii) $\overline{\mathcal{S}}$ contains no zero divisors,

then there is a diagonal unitary matrix D such that

$$\lambda_S D^{-1} S D = |S|$$

for every $S \in \mathcal{S}$. $|\lambda_S| = 1 \ \forall S \in \mathcal{S}$.

None of the conditions are unnecessary.

For example, consider (iii):

Let $E = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$, $F = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

Choose α, β in $(0,1)$ with

$$\alpha^n \neq \beta^m \text{ for all } m, n \in \mathbb{N}.$$

Consider the positive-homogeneous semigroup $\mathcal{S}$ generated by

$$S = E + \alpha F \quad \text{and} \quad T = F + \beta E.$$

A typical member of $\mathcal{S}$ is

$$S^n T^m = \frac{1}{2}\begin{pmatrix} \beta^m + \alpha^n & \beta^m - \alpha^n \\ \beta^m - \alpha^n & \beta^m + \alpha^n \end{pmatrix}.$$

Every member is indecomposable.

$\beta^m - \alpha^n > 0 \Rightarrow S^n T^m \geq 0$

$\beta^m - \alpha^n < 0 \Rightarrow D^{-1}(S^n T^m) D \geq 0$ with $D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\rho(S^n T^m) = \rho(|S^n T^m|)$.

Another example to show that indecomposability of $\mathcal{S}$ as a whole is not enough:

Let $P = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdot & \cdot & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$.

Then

$$\{ \alpha D P^k : k \in \mathbb{N}, \alpha \in \mathbb{R}^+, D \text{ diagonal} \}$$

is an indecomposable, positive-homogeneous semigroup with no zero divisors.

Note $\overline{\mathcal{S}} = \mathcal{S}$, and $\rho(S) = \rho(|S|)$ for all $S \in \mathcal{S}$. <u>But</u> there are decomposable members. $\mathcal{S}$ cannot be positivsed:

Just think of all the diagonal members!

Some points of the proof of Theorem 1:

For $S \neq 0$ in $\mathcal{S}$, the Perron-Frobenius idempotent

$$E = \lim_m \left(\frac{|S|}{\rho(S)} \right)^{rm}$$

has to have rank $r = 1$. Otherwise, $|S|^r$ and thus S^r are block-diagonal $\Rightarrow$ S^r is decomposable, contradiction.

Use extended Wielandt result on S:

$\exists$ D with $|S| = \lambda_S D S D^{-1}$. So

$$F = \lim_m \left(\frac{S}{\rho(S)} \right)^{rm}$$

is also rank-one. Show that it is indecomposable.

Easily verified: the Wielandt D for F is the same[††] as that for S itself.

†† okay, except for a scalar coefficient.

So the problem of showing that the D_α for all S_α' are the same (which completes the proof) reduces to the case where the rank is one. Here we use the hypothesis $EF \neq 0$, $FE \neq 0$ (where E and F are the rank-one idempotents in $\overline{\mathcal{S}}$).

Let us look at the (quite typical) finite-dimensional case:

All entries of E and F are nonzero by indecomposability. Use Wielandt's theorem on E: get diagonal D such that $D^{-1}ED = |E|$. Apply $D^{-1}(\cdot)D$ to all of $\mathcal{S}$ to assume with no loss of generality that $E \geq 0$.

so $E = uv^*$, $F = xy^*$, where

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \quad v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \text{ and } y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

and $u_i > 0$, $v_i > 0$, $|x_i| > 0$, $|y_i| > 0$ $\forall i$

No zero divisors $\Longrightarrow$ $v^* x \neq 0$, $y^* u \neq 0$.

Now $|\mathrm{tr}\, uv^*xy^*| = |\mathrm{tr}(EF)| = \rho(EF) = \rho(|EF|)$

$$= \mathrm{tr}(|EF|) = \mathrm{tr}(|uv^*xy^*|).$$

so $|v^*x| \cdot |\mathrm{tr}\, uy^*| = |v^*x|\, \mathrm{tr}\, |uy^*|$

or $|\mathrm{tr}\, uy^*| = \mathrm{tr}\, |uy^*|$.

This means $\left|\sum_{i=1}^{n} u_i \bar{y}_i\right| = \sum_{i=1}^{n} u_i |y_i|$,

which implies that $\exists$ a with $y = a|y|$.

Similarly, using FE instead of EF, we get b with $x = b|x|$ and $xy^* = ab|x| \cdot |y|^*$.

Thus F is already effectvely positive. $\square$

What about the more desirable hypothesis of indecomposability for $\mathcal{S}$ (and not for each member)?

<u>Theorem 2.</u> If $\mathcal{S} = \overline{\mathcal{S}}$ is a convex, positive-homogeneous, indecomposable $\frac{1}{2}$group of regular operators such that

(1) $|S|$ is compact and $\rho(S) = \rho(|S|)$ for all $S \in \mathcal{S}$, and

(2) $\mathcal{S}$ has no zero divisors, then there exists a unitary M_φ, $\varphi \in \mathcal{L}^\infty(X, \mu)$, such that for all $S \in \mathcal{S}$

$$S = \lambda_S \, M_\varphi^{-1} \, |S| \, M_\varphi$$

with some $\lambda_S \in \mathbb{T}$.

Indication of proof for the discrete case $\ell^2(\mathbb{N})$ or $\mathbb{C}^n$

First,

<u>an easy lemma</u> needed in the proof:

If $\mathcal{S}$ is a closed convex, positive-homogeneous set of operators on $\ell^2(\mathbb{N})$, then there is $T \in \mathcal{S}$ with

$$\operatorname{supp}(T) = \bigcup_{S \in \mathcal{S}} \operatorname{supp} S.$$

(The (i,j) entry in the matrix of T is zero if and only if every $S \in \mathcal{S}$ has its (i,j) entry $= 0$.)

Five steps in the proof:

(1) By the lemma, $\exists$ an indecomposable member S_0 with maximal support.

(2) These members form a dense set in $\mathcal{S}$:

Given $S \in \mathcal{S}$ and $\varepsilon > 0$, the uncountable set $\{S + \lambda S_0 : |\lambda| < \varepsilon\}$ contains a member with support equal to $\operatorname{supp} S_0$.

(3) The Perron–Frobenius idempotents of these members have rank one — this time, because we look at the block-monomial form, and if $r > 1$, then $T^2 \neq 0$

$$\operatorname{supp} T^2 \not\subseteq \operatorname{supp} T,$$

a contradiction. So $r = 1$.

(4) These P.-F. idempotents generate a subsemigroup $\mathcal{S}_1$ of $\mathcal{S}$ in which every member has rank one. (There are no zero divisors in $\mathcal{S}$.) So Theorem 1 is applicable to $\mathcal{S}_1$: One fixed M_φ does the job for all of $\mathcal{S}_1$.

(5) Every member of the dense set of maximal-support members of $\mathcal{S}$ has the same M_φ as its P.-F. idemptent (in $\mathcal{S}_1$)

$\therefore$ $\mathcal{S}$ is positivised by M_φ. $\square$