Complete decomposability of positive compact operators with positive commutators

Joint work with Roman Drnovšek

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A positive operator is an operator which maps almost everywhere positive functions to almost everywhere positive functions. An operator A is of constant sign if A is positive or $-A$ is positive. Operators A and B semi-commute if $AB - BA \geq 0$ or $BA - AB \geq 0$, i.e., the commutator $AB - BA$ is of constant sign.

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If $a \subseteq X$ is an atom of measure and P_a is the projection to the one-dimensional standard subspace $L^p(a)$, the atomic diagonal of the positive operator T is

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where $\mathcal{F}$ runs over all finite sets consisting of all pairwise disjoint atoms of X. A number λ is contained on the diagonal of the operator T if $\lambda = \mathrm{tr}(P_a T P_a)$ for some atom a in X.

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What can be said in the infinite dimensional case?

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The following statements are equivalent for a positive trace class operator T on the space $L^p(X)$ $(1 \leq p < \infty)$.

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Theorem (Drnovšek, Kandić 20)**

Previous theorem also holds for positive power compact operators.

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It can be easily verified that $\mathcal{D}(S)\mathcal{D}(T) = \mathcal{D}(T)\mathcal{D}(S)$ for all positive operators S and T. Therefore, $\mathcal{D}(ST) = \mathcal{D}(S)\mathcal{D}(T)$ for all $S, T \in \mathcal{S}$ implies

$$\mathcal{D}(ST) = \mathcal{D}(TS)$$

for all $S, T \in \mathcal{S}$. Is this enough for complete decomposability?

Theorem (Drnovšek, Kandić 201*)

A semigroup $\mathcal{S}$ of completely decomposable positive compact operators with the property that for every $T, S \in \mathcal{S}$ we have $\mathcal{D}(ST) = \mathcal{D}(TS)$ is completely decomposable.

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Example

$$A = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

Then $A^2 = B^2 = AB = BA = 0$, so that $\mathcal{S} = \{0, A, B\}$ is a semigroup of completely decomposable matrices such that $\mathcal{D}(S) = 0$ for all $S \in \mathcal{S}$. However, $\mathcal{S}$ is not completely decomposable, as the diagonal of the matrix

$$|A| + B = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

is zero, but the matrix is not nilpotent.

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- Then for all A, B in $\mathcal{S}$ the commutator $AB - BA$ is quasinilpotent.
- $\mathcal{S}$ is triangularizable.

Theorem (Drnovšek, Kandić 20)**

Let $\mathcal{S}$ be a semigroup of completely decomposable constant sign compact operators. If the diagonal of the commutator $AB - BA$ is of constant sign for all $A, B \in \mathcal{S}$, then $\mathcal{S}$ is completely decomposable.

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Corollary

A commutative semigroup of completely decomposable positive compact operators is completely decomposable.

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Theorem (Drnovšek, Kandić 20**)

Let A and B be completely decomposable positive power compact operators. If $AB \geq BA$, then A and B are simultaneously completely decomposable.

Corollary

Let $\mathcal{F}$ and $\mathcal{G}$ be families of positive compact operators. Suppose that $\mathcal{F}$ and $\mathcal{G}$ are completely decomposable. If $[A, B] \geq 0$ for all $A \in \mathcal{F}$ and $B \in \mathcal{G}$, then $\mathcal{F} \cup \mathcal{G}$ is completely decomposable.

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Theorem (Radjavi 1987)

A set $\mathcal{N}$ of nilpotent endomorphisms of a finite dimensional vector space is triangularizable whenever for $A, B \in \mathcal{N}$ there exists a noncommutative polynomial $p(A, B)$ such that $AB + p(A, B)A \in \mathcal{N}$.

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- In general, the answer is no! (Read's operator);
- Yes in the case of algebras of quasinilpotent compact operators (Shulman, Turovskii 2000);
- A Lie algebra $\mathcal{L}$ of compact operators is triangularizable if the spectrum is subadditive on $\mathcal{L}$ (Kennedy, Radjavi 2009) and (Shulman, Turovskii 2005).

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2. *Every pair of operators from $\mathcal{M}$ is triangularizable;*
3. *For all A, B and C in $\mathcal{M}$ the operator $A(BC - CB)$ is quasinilpotent.*

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Let $\mathcal{H}$ be a Hilbert space and $\mathcal{M}$ a Lie set in $\mathcal{B}^1(\mathcal{H})$. The following statements are equivalent:

1. *$\mathcal{M}$ is triangularizable;*
2. *Every pair of operators from $\mathcal{M}$ is triangularizable;*
3. *For all A, B and C in $\mathcal{M}$ the operator $A(BC - CB)$ is quasinilpotent.*

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- The trace condition $\mathrm{tr}(ABC) = \mathrm{tr}(ACB)$ on algebra of trace class operators implies its triangularizability.

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- Let $\mathcal{M}$ be a Lie set of completely decomposable compact operators of constant sign. Is $\mathcal{M}$ completely decomposable?

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