talk for LAW2014, Ljubljana, Slovenia, June 2014

Speaker: John Holbrook, University of Guelph, Ontario, Canada

Abstract: The recent paper [CGH2014] completes work begun some years ago with Frank Gilfeather at the Maui High Performance Computing Center (MHPCC, located on the Hawaiian island of Maui). It combines that work with important new ideas due to Michel Crouzeix. We explain how the problem treated in [CGH2014] developed from a highly influential 1970 paper by Paul Halmos, which drew attention to ten research problems about Hilbert space operators. Among the most stimulating was the following: find an intrinsic property of an operator T that holds iff T is similar to a contraction C. Halmos proposed that such a property might be: $K(T) < \infty$, where $K(T)$ is the so-called polynomial bound of T, ie the supremum of $\|p(T)\|$ over polynomials p mapping the unit disc into itself.

Many important tools were developed in response to this problem, notably by Arveson, Paulsen, Bourgain, Pisier, and Davidson. Pisier finally (c.1995) showed that the Halmos criterion must be strengthened. We'll give an account of these developments (suitable for a general mathematical audience) leading up to the related puzzle resolved in our joint work [CGH2014].

Many important tools were developed in response to this problem, notably by Arveson, Paulsen, Bourgain, Pisier, and Davidson. Pisier finally (c.1995) showed that the Halmos criterion must be strengthened. We'll give an account of these developments (suitable for a general mathematical audience) leading up to the related puzzle resolved in our joint work [CGH2014].

[CGH2014] M. Crouzeix, F. Gilfeather, and J. Holbrook, Polynomial bounds for small matrices, Linear and Multilinear Algebra, DOI: 10.1080/03081087.2013.777439, 12 pages, 2014

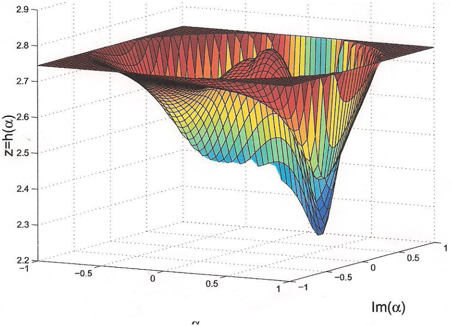

Figure 1: Here $z = h(\alpha) = \max\{\|\mu_\alpha(T)\mu_\beta(T)\| : \beta \in \overline{\mathbb{D}}\}$. Our thanks to Ken Davidson who noted that we have unconsciously chosen a graphic resembling the crater of Haleakala on the island of Maui, where some of this work was done. The $T \in \mathcal{G}_3$ chosen here has μ–index 1, because the interior peak is at the same height as the crater rim.

Here we speak of Hilbert space **operators** T, ie linear $T : X \to X$ where X is a complex Hilbert space. In the finite n–dimensional case, we can think of $X = \mathbb{C}^n$ and represent T by an $n \times n$ complex matrix. For the infinite–dimensional case it's enough to think of the sequence analog $X = \ell^2(\mathbb{C})$.

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Our operators are **bounded**, ie $\|T\| < \infty$ where

$$\|T\| = \sup\{\|Tu\| : u \in \mathbb{C}^n, \|u\| = 1\},$$

and $\|x\|$ is the Hilbert space (Euclidean) norm of $x \in X$.

Note that $\|T\|$ is the Lipschitz constant of the map $T : X \to X$, so that

$$\|TS\| \leq \|T\|\|S\|.$$

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In particular, if T is similar to a contraction C ie

$$T = SCS^{-1}, \quad \|C\| \leq 1,$$

then for $k = 0, 1, 2, \ldots$

$$\|T^k\| = \|SC^kS^{-1}\| \leq \|S\|\|C\|^k\|S^{-1}\| \leq \|S\|\|S^{-1}\|,$$

the **condition number** of S.

Thus T similar to a contraction implies that $K_0(T) < \infty$, where

$$K_0(T) = \sup_{k \geq 0} \|T^k\|.$$

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This suggestion was perhaps due to Sz–Nagy, in view of his beautiful 1947 result:

T is similar to a unitary operator U (both U and U^{-1} are contractions) **iff**

$$\sup\{\| T^k \| : -\infty < k < \infty\} < \infty.$$

The necessity of this condition is clear enough; on the other hand, Sz–Nagy's proof of sufficiency was a brilliant gem of classic functional analysis: define

$$< h, g >= LIM\{\frac{1}{m}\sum_{k=1}^{m}(T^k h, T^k g)\},$$

where (h, g) is the inner product of $h, g \in X$ and LIM is a **Banach limit**, ie a norm–preserving linear extension of the limit for convergent sequences to the space of all bounded complex sequences (Hahn–Banach Theorem, so a bit of transfinite sorcery).

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The introduction of Cesaro averages above makes the expression translation invariant on sequences so that $< Th, Tg >=< h, g >$. Let

$$M = \sup\{\|T^k\| : -\infty < k < \infty\}.$$

We have $(1/M^2)(h, h) \leq < h, h > \leq M^2(h, h)$, so that $< \cdot, \cdot >$ is a new inner product equivalent to the original.

The relation $< Th, Tg >=< h, g >$ says that T is unitary in the new geometry. If we want to obtain the similarity explicitly we may write

$$< h, g >= (Ah, g) = (Bh, Bg),$$

where A is a positive operator and $B = A^{\frac{1}{2}}$; then

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It's not hard to show that $K_0(T) < \infty$ implies that T is similar to a contraction in the finite–dimensional case. A useful idea here is that if all eigenvalues of T are in the open unit disc, so that the spectral radius $r(T) < 1$, then the spectral radius formula

$$r(T) = \lim \|T^k\|^{\frac{1}{k}}$$

implies that

$$\sum_{k=0}^{\infty}(T^k h, T^k g)$$

is convergent, defining a new inner product $< h, g >$ such that $< Th, Th > \leq < h, h >$, ie T is a contraction with respect to the new inner product.

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In the infinite–dimensional case, however, Foguel (1964) found examples where $K_0(T) < \infty$ but T is **not** similar to a contraction.

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A more demanding condition on operators similar to contractions follows from **von Neumann's inequality**:

in a remarkable 1951 paper, von Neumann showed that for a Hilbert space contraction C and any polynomial $p(z)$

$$\|p(C)\| \leq \|p\|_\infty =_{definition} \max\{|p(z)| : z \in \mathbb{C},\ |z| \leq 1\}.$$

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Later we'll give a simple proof of this and much more.
Thus, if $T = SCS^{-1}$ we have

$$K(T) =_{definition} \sup\{\|p(T)\| : \|p\|_\infty \leq 1\} \leq \|S\|\|S^{-1}\|.$$

In fact, $K(T) \leq M(T)$, where

$$M(T) = \inf\{\|S\|\|S^{-1}\| : \|S^{-1}TS\| \leq 1\}.$$

The question that Halmos promoted in 1970 (and which took 25 years to answer) can now be expressed as follows:

$$K(T) < \infty \quad \Longrightarrow \quad M(T) < \infty?$$

Meanwhile Arveson was developing the notions of **complete positivity** and **completely contractive maps**, which were crucial for the study of **operator algebras** and more recently for **quantum information theory** (for example, the standard model for a **quantum channel** is a completely positive map between quantum systems).

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$\|T\|_{cb} < \infty$, where $\|T\|_{cb} = \sup_m \|\varphi_m\|$ and φ_m is the map on $m \times m$ matrices of polynomials defined by:

$$\varphi_m([p_{ij}]) = [p_{ij}(T)],$$

as an $m \times m$ block matrix acting on the Hilbert space X^m.

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This condition clearly comes from the fact that when T is replaced by a contraction C the maps φ_m are contractive (the **completely contractive** property; for von Neumann's inequality take $m = 1$).

In [N1961] Edward Nelson gave a very simple proof of von Neumann's inequality (later noticed also by Pisier); it is based on nothing more than a special case of the maximum principle from complex analysis: if p is a polynomial, then $\max\{|p(z)| : |z| \leq 1\}$ (ie $\|p\|_\infty$) occurs on the boundary (unit circle).

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[N1961] E. Nelson, The distinguished boundary of the unit operator ball, Proc. Amer. Math. Soc. 12, 994–5, 1961

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Note that
$\|[p_{ij}(C)]\| = \max\{|([p_{ij}(C)]u, w)| : u, w \in X^m, \|u\| \leq 1, \|w\| \leq 1\}$.
Express C via its singular value decomposition, $C = UDW$, where U and W are unitary, $D = \mathrm{diag}(s_1, \ldots, s_n)$, and s_k are the singular values of C. Thus for each fixed u, w

$$([p_{ij}(C)]u, w) = P(s_1, \ldots, s_n),$$

where P is a certain polynomial in n variables.

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$$([p_{ij}(C)]u, w) = P(s_1, \ldots, s_n),$$

where P is a certain polynomial in n variables.
Since $\|C\| \leq 1$ all s_k lie in the unit disc (in fact, in $[0, 1]$), the maximum principle implies that

$$|([p_{ij}(C)]u, w)| \leq |P(z_1, \ldots, z_n)|,$$

where each z_k lies on the unit circle.

Let $A = \mathrm{diag}(z_1, \ldots, z_n)$; note that A is unitary and

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Now UAW is unitary (by unconscious choice, UAW is also a union!) so it only remains to show that

$$\|[p_{ij}(B)]\| \leq \max\{\|[p_{ij}(z)]\| : |z| \leq 1\}$$

for unitary B. Write B in diagonal form: $B = \operatorname{diag}(b_1, \ldots, b_n)$, with each $|b_k| = 1$.

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Now each $p_{ij}(B) = \mathrm{diag}(p_{ij}(b_1), \ldots, p_{ij}(b_n))$ so that

$$\|[p_{ij}(B)]\| = \|\bigoplus_{k=1}^{n}[p_{ij}(b_k)]\| = max_k\|[p_{ij}(b_k)]\|.$$

QED

The polynomially bounded condition says that $K(T) = \|\varphi_1\| < \infty$. The condition $\|T\|_{cb} < \infty$ is more complicated but it too can be viewed as an intrinsic condition on T and so as a legitimate answer to the Halmos question. Moreover, Arveson was right, and Vern Paulsen proved it (and more) in two 1984 papers; the story is nicely presented (along with many other matters) in Vern's book (second edition) [Pa2002].

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[Pa84] V. Paulsen, Every completely polynomially bounded operator is similar to a contraction, J. Funct. Anal. 55 (1984)
[Pa84a] V. Paulsen, Completely bounded homomorphisms of operator algebras, Proc. Amer. Math. Soc. 92 (1984)
[Pa2002] V. Paulsen, Completely Bounded Maps and Operator Algebras, Cambridge U. P. (2002)

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For example, suppose there are operators T_n on $\mathbb{C}^n$ such that $K(T_n)$ is bounded but $M(T_n)$ is not (which was eventually found to be the case). Then Halmos's suggestion is wrong: consider

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$$T = \oplus_{n=1}^{\infty} T_n.$$

We have $K(T) = \sup K(T_n) < \infty$, but if $M(T)$ were finite we'd have an inner product norm $|\cdot|$ with respect to which $|T| \leq 1$ and, since $M(T)$ depends only on the equivalence constant relating $|\cdot|$ to $\|\cdot\|$, we would have the contradiction

$$\sup M(T_n) \leq M(T).$$

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Again, constructing the right new inner product was a key element. The method came from the older paper [Ho1971], based in part on and even older observation of Jordan and von Neumann: one can first define an appropriate equivalent **norm** on X, then establish that it's an inner product norm by verifying the parallelogram law.

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[Ho1971] J. Holbrook, Spectral dilations and polynomial bounded operators, Indiana University Math. J. 20, No. 11 (1971)
[J–vN1935] P. Jordan and J. von Neumann, On inner products on linear metric spaces, Ann. Math. 36 (1935)

We've seen that the Halmos problem is closely related to a finite–dimensional problem:

how is $M(T)$ controlled by $K(T)$ for operators T on $\mathbb{C}^n$ ($n \times n$ matrices)?

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Jean Bourgain, in a 1986 paper, used Paulsen's relation $M(T) = \|T\|_{cb}$ and some fancy function theory to obtain a bound on $M(T)$ in terms of $K(T)$ and a slowly growing factor depending on the underlying dimension n:

$$M(T) \leq cK^4(T) \log n,$$

where c is a universal constant.

Finally, c.1995, Gilles Pisier answered the Halmos question in the negative, showing (among other things) that there exist n–dimensional operators T_n such that $K(T_n)$ is bounded while $M(T_n) \geq K\sqrt{\log n}$ for some constant K.

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What is the true (maximal) growth rate of $M(T_n)$?

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What is the true (maximal) growth rate of $M(T_n)$?

Pisier's argument was somewhat mysterious and depended in part on deep probabilistic methods. Davidson and Paulsen, however, soon offered an alternate approach that is simpler and more direct.

The papers:

[Pi96] G. Pisier, A polynomially bounded operator on Hilbert space which is not similar to a contraction, C. R. Acad. Sci. Ser 1 Math. 322 (1996), pp.547–550

[Pi97] G. Pisier, A polynomially bounded operator on Hilbert space which is not similar to a contraction, J. Amer. Math. Soc. 10 (1997), pp.351–369.

[D–P97] K. Davidson and V. Paulsen, On polynomially bounded operators, J. für die reine und angewandte Math. (1997)

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It was noted early on that $M(T) = K(T)$ for **2x2 matrices**.

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It was noted early on that $M(T) = K(T)$ for **2x2 matrices**.

My first proof was a not–very–enlightening calculation; presently we'll show the 2x2 result in the light of more general principles.

First of all, one clearly needed to explore the possibility that $M(T) = K(T)$ for **all** dimensions.

Pisier–Davidson–Paulsen eventually showed that this cannot be the case, but that work seems of little help in constructing simple matrices T that exhibit $K(T) < M(T)$.

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Consider T on finite–dimensional X ($\equiv \mathbb{C}^n$) with $M(T) < \infty$. Let $\| \cdot \|$ denote the (inner product) norm on X and $| \cdot |$ a new (ip) norm such that $|T| \leq 1$.

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Let U be the unit sphere in X with respect to $|\cdot|$:

$$U = \{u \in X : |u| = 1\},$$

and let

$$m = \min_{u \in U} \|u\|, \quad M = \max_{u \in U} \|u\|, \quad \delta = M/m.$$

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Then $M(T) = \min\{\delta : |T| \leq 1\}$.

We say T has **big spectrum** if T has distinct eigenvalues $z_1, \ldots, z_n$ and all $|z_k| = 1$. (Actually for many of our results we can allow one z_k to be strictly within the unit disc ...)

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Clearly T with big spectrum is **cryptounitary**, ie similar to a unitary.

Proposition: For such T we have $M(T) \leq K_0^2(T) \leq K^2(T)$.

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The proof is essntially an adaptation of the Sz.–Nagy renorming idea combined with the "almost periodicity" of finite–dimensional unitaries (there are large N for which $T^N \approx I$).

Thus cryptounitaries cannot resolve the Halmos question although, as it turns out, they **can** display $M(T) > K(T)$.

Let $u_1, \ldots, u_n$ be eigenvectors of T corresponding to $z_1, \ldots, z_n$, with $|u_k| = 1$. If $< \cdot, \cdot >$ denotes the ip corresponding to $| \cdot |$, the u_k are orthonormal with respect to $< \cdot, \cdot >$.

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Given $u \in U$ let

$$s(u) = (| < u, u_1 > |^2, \ldots, | < u, u_n > |^2)$$

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Let

$$U_* = \{u \in U : \|u\| = m \text{ (minimal)}\}$$

and

$$U^* = \{u \in U : \|u\| = M \text{ (maximal)}\}.$$

Proposition: If T has big spectrum,

(a) $M(T) = \delta$ **iff** $\mathrm{conv}(s(U_*)) \cap \mathrm{conv}(s(U^*)) \neq \emptyset$;

(b) $K(T) = \delta$ **iff** $s(U_*) \cap s(U^*) \neq \emptyset$.

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The geometric distinction between these two conditions allows construction of examples where $M(T) > K(T)$:

earlier with $n = 12$ (ad hoc),

later with $n = 4$ (interpenetrating ellipsoids ...).

The 4–dimensional examples arise as follows:

if U^* and U_* are "two–dimensional" then $s(U^*)$ and $s(U_*)$ are hollow ellipsoids inside the 3–dimensional simplex (tetrahedron).

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If they interpenetrate we can chose some $u \in U_*$ such that $s(u)$ is strictly interior to $s(U^*)$ and modify $\|\cdot\|$ so that $U_* = \{e^{i\theta}u\}$. Then $s(U_*) = \{s(u)\}$ **does not** intersect with $s(U^*)$ but **does** lie in $\mathrm{conv}(s(U^*))$.

"video proof:"

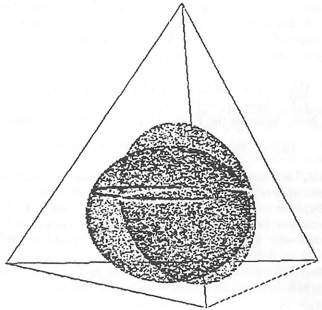

FIG. 1. Interpenetration of "dual" ellipsoids E and E' inside the tetrahedron $\mathrm{conv}(e_1, e_2, e_3, e_4)$. The highlighted rings show horizontal cross sections at $x_1 = 0.3$. See the proof of Theorem 3.4 for more information.

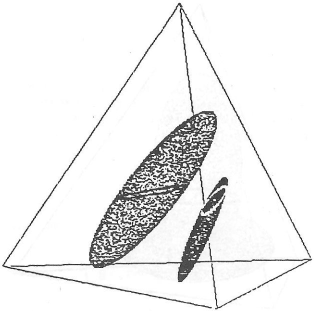

FIG. 2. Disjointness of "dual" ellipsoids; cf. Figure 1.

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[Ho78] J. Holbrook, Distortion coefficients for crypto–unitary operators, Linear Alg. Appl. 19 (1978), pp.189–205
[Ho95] J. Holbrook, Interpenetration of ellipsoids and the polynomial bound of a matrix, Linear Alg. Appl. 229 (1995), pp.151–166

All this suggests we focus on $T \in \mathcal{G}_3$, the space of 3×3 matrices with distinct eigenvalues all lying strictly inside the unit disc. Versions of the following propositions are available for $\mathcal{G}_n$ (any finite n) but here we often look at the simplified forms for $n = 3$ or $n = 2$.

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Given $\alpha \in \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, let $\mu_\alpha : \mathbb{D} \to \mathbb{D}$ denote the Möbius transformation defined by

$$\mu_\alpha(z) = \frac{z - \alpha}{1 - \overline{\alpha} z}.$$

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Proposition 1: If $T \in \mathcal{G}_3$ then

$$K(T) = \max\{\| \prod_{k=1}^{m} \mu_{\alpha_k}(T)\| : \alpha_k \in \mathbb{D}, m < 3\}.$$

This proposition makes the estimation of $K(T)$ computationally feasible, since we need only check finite Blaschke products of length 1 or 2.

The proof is based on the fact that a Blaschke product of length 3 can match the action of any polynomial $p : \mathbb{D} \to \overline{\mathbb{D}}$ at the 3 eigenvalues and that it is even better to expand p to the point where one of the Möbius factors is degenerate.

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Let $\mathcal{B}_m$ denote the class of Blaschke products of length not exceeding m, and, given $T \in \mathcal{G}_n$, let

$$K_m(T) = \max\{\|f(T)\| : f \in \mathcal{B}_m\}.$$

As in Proposition 1, $K(T) = K_{n-1}(T)$ for each $T \in \mathcal{G}_n$, but it may well be the case that $K(T) = K_m(T)$ for some smaller m. Let us call the smallest m such that $K(T) = K_m(T)$ the μ–index of T. Thus, for $n = 3$ the matrices either have μ–index 1 or 2.

In the 3×3 case, Proposition 1 suggests a reasonably efficient (two parameter) algorithm for estimating $K(T)$, since

$$K(T) = \max\{\|\mu_\alpha(T)\mu_\beta(T)\| : \alpha, \beta \in \overline{\mathbb{D}}\},$$

where $\mu_\beta(T)$ is interpreted as $-\beta I$ when $|\beta| = 1$. Given a (randomly chosen) $T \in \mathcal{G}_3$, for each $\alpha \in \overline{\mathbb{D}}$ the value of

$$h(\alpha) = \max_{\beta \in \overline{\mathbb{D}}} \|\mu_\alpha(T)\mu_\beta(T)\|$$

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The observation that $K(T)$, the largest value of $h(\alpha)$, occurs on the "rim of the crater" where $|\alpha| = 1$ indicates that $K(T) = \|\mu_\beta(T)\|$ for some $\beta \in \mathbb{D}$, ie that this T has μ–index 1.

In some other cases, two interior maxima for h are observed in the plot, indicating a T with μ–index 2.

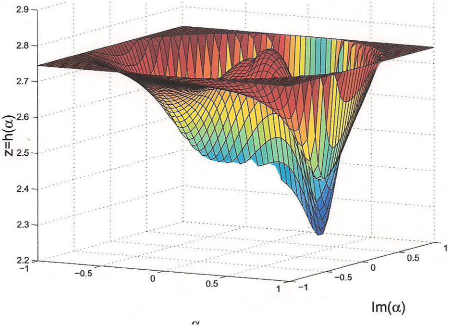

Figure 1: Here $z = h(\alpha) = \max\{\|\mu_\alpha(T)\mu_\beta(T)\| : \beta \in \overline{\mathbb{D}}\}$. Our thanks to Ken Davidson who noted that we have unconsciously chosen a graphic resembling the crater of Haleakala on the island of Maui, where some of this work was done. The $T \in \mathcal{G}_3$ chosen here has μ–index 1, because the interior peak is at the same height as the crater rim.

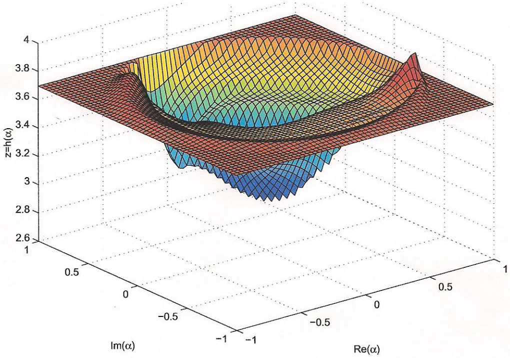

Figure 2: Here again $z = h(\alpha) = \max\{\|\mu_\alpha(T)\mu_\beta(T)\| : \beta \in \overline{\mathbb{D}}\}$. Two interior peaks, higher than the rim, reveal that the $T \in \mathcal{G}_3$ chosen here has μ–index 2.

Proposition 2: Suppose that $T \in \mathcal{G}_n$ and $f \in \mathcal{B}_m$ is such that $\|f(T)\| = K(T)$. Let $g : \mathbb{D} \to \overline{\mathbb{D}}$ be analytic and let $Z = g(T)$ and $Y = f(T)$. If u is a norming vector for Y, then $|(Zu, u)| \leq 1$. In particular, $|(Y^k u, u)| \leq 1$ for every integer $k \geq 2$.

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In order to determine whether or not $M(T) = K(T)$ we use propositions of the following sort. We give explicitly only the case where $n = 2$, but similar criteria apply whenever the μ–index is 1.

Proposition 3: If $T \in \mathcal{G}_n$ has μ–index 1, then $M(T) = K(T)$ **iff** $\|Q\| \le 1$ for all Q in an explicit parametric family of matrices. In the case $n = 2$

$$Q = \begin{bmatrix} 0 & (Y^2u, u) \\ 1 & 0 \end{bmatrix}.$$

Now the fact that $M(T) = K(T)$ when T is 2×2 may be seen as a combination of three more general principles: Proposition 1 (T has μ–index 1), Proposition 2 ($|(Y^2 u, u)| \leq 1$), and Proposition 3 ($\|Q\| \leq 1$).

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When $n = 3$, the family of Q has one real parameter and one can check numerically that $\|Q\| > 1$ sometimes occurs. In this way Frank and I became convinced that $M(T) > K(T)$ can happen for 3×3 T, although we did **not** complete a proper error analysis.

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[G–H00] F. Gilfeather and J. Holbrook, Polynomial bounds for matrices, HPCERC Technical Report HPCERC2000-001 (2000), available at www.carc.unm.edu

The work of Michel Crouzeix:

Michel independently developed many of the same tools as we had done but then applied them and **a tour de force of epsilonics** to show that for sufficiently small ϵ

$$T_\epsilon = \begin{bmatrix} 0 & 2 & 0 \\ \epsilon & 0 & \eta \\ 0 & 0 & \eta \end{bmatrix} \text{ where } \eta = \frac{1-\epsilon^2}{\sqrt{2}}$$

satisfies $K(T_\epsilon) = \|T_\epsilon\| = 2 < M(T_\epsilon)$. (Note that μ–index is 1.)

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[CGH2014] M. Crouzeix, F. Gilfeather, and J. Holbrook, Polynomial bounds for small matrices, Linear and Multilinear Algebra, DOI: 10.1080/03081087.2013.777439, 12 pages, 2014

Wrapping up: the phenomenon $K(T) < M(T)$, which leads to the answer to the Halmos question, begins at $n = 3$ and becomes more and more pronounced as $n \to \infty$ at a rate not entirely determined but somewhere between $\sqrt{\log n}$ and $\log n$.