An elementary proof of Wigner's theorem on quantum mechanical symmetry transformations

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Introduction

Wigner's theorem

Theorem

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{S}$ denote the set of unit vectors of $\mathcal{H}$. Let us consider an arbitrary mapping $\phi\colon \mathcal{S} \to \mathcal{S}$ such that the following holds:

$$|\langle \vec{u}, \vec{v} \rangle| = |\langle \phi(\vec{u}), \phi(\vec{v}) \rangle| \quad (\|\vec{u}\| = \|\vec{v}\| = 1). \tag{W's C}$$

Then there exists a linear or a conjugatelinear isometry $\mathbf{W}\colon \mathcal{H} \to \mathcal{H}$ and a function $f\colon \mathcal{S} \to \mathbb{T} := \{z \in \mathbb{C}\colon |z| = 1\}$ such that we have

$$\phi(\vec{u}) = f(\vec{u}) \cdot \mathbf{W}\vec{u}$$

is satisfied for every unit vector $\vec{u} \in \mathcal{H}$.

Remark: Originally Wigner assumed bijectivity of ϕ.

E. P. Wigner (1931) $\longrightarrow$ the proof was not complete.

J. A. Lomont and P. Mendleson (1963) $\longrightarrow$ first known proof for the classical (i.e. bijective) case.

V. Bargmann (1964) $\longrightarrow$ His proof follows the thoughtline suggested by Wigner.

Figure: Valentine Bargmann

U. Uhlhorn (1963) $\longrightarrow$ A nice generalization ($\dim \mathcal{H} \geq 3, \phi$ is bijective and preserves orthogonality in both directions).

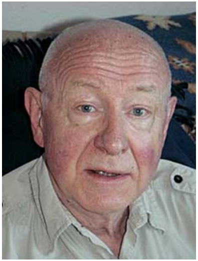

Figure: Ulf Uhlhorn

L. Molnár (1996) $\longrightarrow$ algebraic approach, he managed to generalize Wigner's theorem in several ways.

Figure: Molnár Lajos

$\mathcal{P}_1 := \mathcal{P}_1(\mathcal{H}) \equiv$ the set of rank-one (self-adjoint) projections.
If $\|\vec{u}\| = 1$, then
$\mathbf{P}[\vec{u}] \equiv$ the rank-one projection with precise range $\mathbb{C} \cdot \vec{u}$.

$$|\langle \vec{u}, \vec{v} \rangle|^2 = \underbrace{\operatorname{Tr} \mathbf{P}[\vec{u}]\mathbf{P}[\vec{v}]}_{\text{transition probability}} .$$

$$\|\mathbf{P}[\vec{u}] - \mathbf{P}[\vec{v}]\| = \sqrt{1 - |\langle \vec{u}, \vec{v} \rangle|^2}.$$

Wigner's theorem, two reformulations

Theorem

Let us consider an arbitrary mapping $\phi\colon \mathcal{P}_1 \to \mathcal{P}_1$ *for which the following holds*

$$\operatorname{Tr} \mathbf{P}[\vec{u}]\mathbf{P}[\vec{v}] = \operatorname{Tr} f(\mathbf{P}[\vec{u}])f(\mathbf{P}[\vec{v}]) \quad (\|\vec{u}\| = \|\vec{v}\| = 1). \qquad \text{(W's C)}$$

Then there is a linear or antilinear isometry $\mathbf{W}\colon \mathcal{H} \to \mathcal{H}$ *such that*

$$\phi(\mathbf{P}[\vec{u}]) = \mathbf{W}\mathbf{P}[\vec{u}]\mathbf{W}^* = \mathbf{P}[\mathbf{W}\vec{u}] \quad (\|\vec{u}\| = 1).$$

Theorem

For every isometry $\phi\colon \mathcal{P}_1 \to \mathcal{P}_1$ *there exists a linear or an antilinear isometry* $\mathbf{W}\colon \mathcal{H} \to \mathcal{H}$ *such that we have*

$$\phi(\mathbf{P}[\vec{u}]) = \mathbf{W}\mathbf{P}[\vec{u}]\mathbf{W}^* = \mathbf{P}[\mathbf{W}\vec{u}] \quad (\|\vec{u}\| = 1).$$

Resolving sets

Definition (S. Bau and A. F. Beardon)

Let (X, d) be a metric space and $D, R \subseteq X$. We say that R is a *resolving set* of D if for every two points $x_1, x_2 \in D$ whenever $d(x_1, y) = d(x_2, y)$ is satisfied for all $y \in R$, we necessarily have $x_1 = x_2$.

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$\dim \mathcal{H} = N \in \mathbb{N} \cup \{\aleph_0\}$, $N > 1$.

We fix an orthonormal base: $\{\vec{e}_j\}_{j=1}^N$.

Let $v_j := \langle \vec{v}, \vec{e}_j \rangle$ denote the jth coordinate of a unit vector $\vec{v}$.

The set

$$D := \{\mathbf{P}[\vec{v}] \colon v_j \neq 0, \ \forall\, j\} \subseteq \mathcal{P}_1$$

is **dense** in $\mathcal{P}_1$.

Lemma

Let $\mathcal{H}$ be an arbitrary separable (finite or infinite-dimensional) Hilbert space. Then the set

$$R = \{\mathbf{P}[\vec{e}_j]\}_{j=1}^{N} \cup \left\{\mathbf{P}[\tfrac{1}{\sqrt{2}}(\vec{e}_j - \vec{e}_{j+1})], \mathbf{P}[\tfrac{1}{\sqrt{2}}(\vec{e}_j + i\vec{e}_{j+1})]\right\}_{1\leq j<N}$$

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resolves D.

Proof. An easy calculation. □

Proof of Wigner's theorem
(in the separable case)

Proof. Let $\mathbf{P}[\vec{f_j}] = \phi(\mathbf{P}[\vec{e_j}])$, then $\{\vec{f_j}\}_{j=1}^N$ is an ONS.

$$\mathcal{H}' := \vee\{\vec{f_j}\}_{j=1}^N.$$

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$$\mathcal{H}' := \vee\{\vec{f_j}\}_{j=1}^N.$$

Set a unit vector $\vec{v} \in \mathcal{H}$ and let $\phi(\mathbf{P}[\vec{v}]) = \mathbf{P}[\vec{w}]$. (W's C) implies

$$|v_j| = |\langle\vec{w}, \vec{f_j}\rangle| \quad (\forall j),$$

and from Parseval's identity we get $\vec{w} \in \mathcal{H}'$.

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We define a linear isometry

$$\mathbf{V} \colon \mathcal{H} \to \mathcal{H}' \subseteq \mathcal{H}, \quad \mathbf{V}\vec{e_j} = \vec{f_j} \quad (j \in \mathbb{N}_N).$$

The mapping $\phi_1(\cdot) := \mathbf{V}^* \phi(\cdot) \mathbf{V}$ satisfy (W's C).

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$$\phi_1(\mathbf{P}[\vec{e_j}]) = \mathbf{V}^*\phi(\mathbf{P}[\vec{e_j}])\mathbf{V} = \mathbf{V}^*\mathbf{P}[\vec{f_j}]\mathbf{V} = \mathbf{P}[\mathbf{V}^*\vec{f_j}] = \mathbf{P}[\vec{e_j}] \quad (\forall\, j).$$

Again from (W's C)

$$\phi_1\big(\mathbf{P}[1/\sqrt{2}\cdot(\vec{e}_j - \vec{e}_{j+1})]\big) = \mathbf{P}[1/\sqrt{2}\cdot(\vec{e}_j - \delta_{j+1}\vec{e}_{j+1})]$$

$$\phi_1\big(\mathbf{P}[1/\sqrt{2}\cdot(\vec{e}_j + i\vec{e}_{j+1})]\big) = \mathbf{P}[1/\sqrt{2}\cdot(\vec{e}_j - \varepsilon_{j+1}\vec{e}_{j+1})]$$

where $|\delta_{j+1}| = |\varepsilon_{j+1}| = 1$ $(1 \leq j < N)$.

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where $|\delta_{j+1}| = |\varepsilon_{j+1}| = 1$ $(1 \leq j < N)$. Therefore

$$\sqrt{2} = |1 + \delta_{j+1}\overline{\varepsilon_{j+1}}|,$$

and consequently $\delta_{j+1} = \pm i\varepsilon_{j+1}$ $(j < \mathbb{N})$.

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and consequently $\delta_{j+1} = \pm i\varepsilon_{j+1}$ $(j < \mathbb{N})$.
We define $\phi_2(\cdot) := \mathbf{U}^*\phi_1(\cdot)\mathbf{U}$, where

1. If $\varepsilon_2 = -i\delta_2$, then let $\mathbf{U}$ be the unitary operator such that

$$\mathbf{U}\vec{e_1} = \vec{e_1}, \quad \mathbf{U}\vec{e_k} = \left(\prod_{j=2}^{k}\delta_j\right)\vec{e_k} \quad (k > 1).$$

2. If $\varepsilon_2 = i\delta_2$, then let $\mathbf{U}$ be the antiunitary operator defined by the equations above.

Moreover,

$$\phi_2(\mathbf{P}[\vec{e}_j]) = \mathbf{P}[\vec{e}_j] \quad (\forall\, j),$$

$$\phi_2(\mathbf{P}[1/\sqrt{2}\cdot(\vec{e}_j - \vec{e}_{j+1})]) = \mathbf{P}[1/\sqrt{2}\cdot(\vec{e}_j - \vec{e}_{j+1})] \quad (\forall\, j),$$

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We also have

$$\phi_2(\mathbf{P}[1/\sqrt{2}\cdot(\vec{e}_j + i\vec{e}_{j+1})]) = \mathbf{P}[1/\sqrt{2}\cdot(\vec{e}_j \pm i\vec{e}_{j+1})] \quad (1 < j < N).$$

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Assume that there exists an index $j > 1$ for which

$$\phi_2(\mathbf{P}[1/\sqrt{2}\cdot(\vec{e}_j + i\vec{e}_{j+1})]) = \mathbf{P}[1/\sqrt{2}\cdot(\vec{e}_j - i\vec{e}_{j+1})]$$

holds. We may assume that this j is the first such index.

Claim Then

$$\phi_2(\mathbf{P}[v_{j-1}\vec{e}_{j-1} + t\vec{e}_j + v_{j+1}\vec{e}_{j+1}]) = \mathbf{P}[v_{j-1}\vec{e}_{j-1} + t\vec{e}_j + \overline{v_{j+1}}\vec{e}_{j+1}]$$

holds for every $t > 0, v_{j-1} \neq 0, v_{j+1} \neq 0, |v_{j-1}|^2 + t^2 + |v_{j+1}|^2 = 1.$

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Now, let

$$\vec{x} = \frac{-1}{2}\vec{e}_{j-1} + \frac{1}{2}\vec{e}_j + \frac{1}{\sqrt{2}}\vec{e}_{j+1}, \quad \vec{y} = \frac{i}{2}\vec{e}_{j-1} + \frac{1}{2}\vec{e}_j + \frac{i}{\sqrt{2}}\vec{e}_{j+1}.$$

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Claim and (W's C) implies

$$\sqrt{2}/4 = |i/4 + 1/4 - i/2| = |i/4 + 1/4 + i/2| = \sqrt{10}/4,$$

which is a contradiction.

Claim Then

$$\phi_2(\mathbf{P}[v_{j-1}\vec{e}_{j-1} + t\vec{e}_j + v_{j+1}\vec{e}_{j+1}]) = \mathbf{P}[v_{j-1}\vec{e}_{j-1} + t\vec{e}_j + \overline{v_{j+1}}\vec{e}_{j+1}]$$

holds for every $t > 0, v_{j-1} \neq 0, v_{j+1} \neq 0, |v_{j-1}|^2 + t^2 + |v_{j+1}|^2 = 1$.

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Claim and (W's C) implies

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which is a contradiction. Therefore ϕ_2 is the identity mapping on $\mathcal{P}_1$, and $\phi(\mathbf{P}[\vec{u}]) = \mathbf{W}\mathbf{P}[\vec{u}]\mathbf{W}^*$ with $\mathbf{W} = \mathbf{V}\mathbf{U}$. □

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- Note that we could consider $\phi\colon \mathcal{P}_1(\mathcal{H}) \to \mathcal{P}_1(\mathcal{K})$ with some Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$.

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- Note that we could consider $\phi\colon \mathcal{P}_1(\mathcal{H}) \to \mathcal{P}_1(\mathcal{K})$ with some Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$.
- The proof is similar (but easier) in real Hilbert spaces.

References 1/2

- E. P. Wigner, *Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektrum*, Fredrik Vieweg und Sohn, 1931.
- J. S. Lomont and P. Mendelson, The Wigner unitary-antiunitary theorem, *Ann. Math.* **78** (1963), 548–559.
- V. Bargmann, Note on Wigner's theorem on symmetry operations, *J. Math. Phys.* **5** (1964), 862–868.
- U. Uhlhorn, Representation of symmetry transformations in quantum mechanics, *Ark. Fysik* **23** (1963), 307–340.

References 2/2

- L. Molnár, Wigner's unitary-antiunitary theorem via Herstein's theorem on Jordan homeomorphisms, *J. Nat. Geom.* **10** (1996), 137–148.
- L. Molnár, An algebraic approach to Wigner's unitary-antiunitary theorem, *J. Austral. Math. Soc.* **65** (1998), 354–369.
- Gy. P. Gehér, An elementary proof for the non-bijective version of Wigner's theorem, to appear in *Phys. Lett. A*.

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