Simultaneously self-adjoint sets of matrices

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Outline

Two questions

(1) Consider a set of matrices $\mathcal{M} \subset \mathbb{C}^{d \times d}$. When are all the elements of $\mathcal{M}$ simultaneously equivalent to hermitian matrices under the natural action of $GL_d(\mathbb{C}) \times GL_d(\mathbb{C})$? In other words, when do there exist $A, B \in GL_d(\mathbb{C})$ such that ANB is hermitian for all $N \in \mathcal{M}$?

(2) Assume that the answer to (1) is positive. Is there an element in $\mathcal{M}$ that is equivalent (under this simultaneous equivalence) to a positive definite matrix? In other words, given a set of hermitian $d \times d$ matrices, when do these matrices admit a positive definite linear combination?

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Approaches

- **linear algebra:** simultaneous reduction of a set of matrices to hermitian (or symmetric) form
- **semidefinite programming:** linear matrix inequality (LMI) representations
- **algebraic geometry:** cubic curves, surfaces and hypersurfaces as zero loci of determinants

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Approaches

Computationally both questions are straightforward:

Question (1) reduces to a system of linear equations over $\mathbb{R}$,

$$CN_i^* = N_i C^*, \quad i + 1, \ldots, n,$$

where $C = A^{-1}B^*$ and $N_1, \ldots, N_n$ is a basis of the $\mathbb{R}-$linear span of $\mathcal{M}$.
Question (2) is solved by semidefinite programming (at least for small d and n).

Simultaneously self-adjoint sets of matrices

Definition

Let $\mathcal{M} \subset \mathbb{C}^{d \times d}$ be a set of square matrices. We call $\mathcal{M}$ simultaneously self-adjoint if there exist invertible $A, B \in GL_d(\mathbb{C})$ such that ANB are complex hermitean matrices for all $N \in \mathcal{M}$.

We can restrict to finite sets:

Lemma

The following statements are equivalent:

- $\mathcal{M}$ *is simultaneously self-adjoint*
- $\mathcal{L}in_{\mathbb{R}}\mathcal{M}$ *is simultaneously self-adjoint.*
- *Any basis of* $\mathcal{L}in_{\mathbb{R}}\mathcal{M}$ *is simultaneously self-adjoint.*

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Definite and indefinite sets of matrices

- A set $\mathcal{M}$ of complex hermitean matrices is definite if there exist $k_0, \ldots, k_n \in \mathbb{R}$ and $M_0, \ldots, M_n \in \mathcal{M}$ such that

$$k_0 M_0 + k_1 M_1 + \cdots + k_n M_n > 0.$$

  It is indefinite otherwise.
- A vector $v \in \mathbb{C}^d$ is self-orthogonal for $\mathcal{M}$ if

$$vNv^* = 0 \quad \text{for all} \quad N \in \mathcal{M}.$$

Note that $\mathcal{M}$ with a self-orthogonal vector is always indefinite.

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Determinantal representations

- Subset $\mathcal{M}$ is regular if it contains an invertible matrix, i.e. $\mathcal{M} \cap \mathrm{GL}_3(\mathbb{C}) \neq \emptyset$.
- To $\mathcal{M}$ with a basis $\{M_0, \ldots, M_n\}$ we assign matrix

$$M(x_0, \ldots, x_n) = x_0 M_0 + x_1 M_1 + \ldots + x_n M_n$$

whose entries are linear in $x_0, \ldots, x_n$. When $\mathcal{M}$ is regular, we call the matrix M a determinantal representation of the hypersurface

$$\{(x_0, \ldots, x_n) \subset \mathbb{P}^n \,;\, \det M(x_0, \ldots, x_n) = 0\}.$$

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Hypersurfaces

- The underlying field is $\mathbb{C}$, often we restrict to $\mathbb{R}$.
- $F(x_0, \ldots, x_n)$ is a homogeneous polynomial of degree $d \geq 2$ in $n+1$ variables.
- The zero locus $\{F(x_0, \ldots, x_n) = 0\} \subset \mathbb{P}^n$ defines a hypersurface in $\mathbb{P}^n$

Example: The Weierstrass cubic curve is defined by

$$\{(x, y, z) \subset \mathbb{P}^2 \; ; \; -y^2 z + x^3 + px^2 z + qxz^2 = 0\}, \quad p, q \in \mathbb{C}.$$

The set of zeros $F(x_0, x_1, x_2, x_3)$ defines a surface in $\mathbb{P}^3$.

Determinantal representations are well-defined.

- Different choices of basis for $\mathcal{M}$ yield projectively equivalent hypersurfaces (linear coordinate change in the determinant polynomials).
- Equivalent determinantal representations $M(x_0, \dots, x_n)$ and $M'(x_0, \dots, x_n) = A\,M(x_0, \dots, x_n)\,B$ for $A, B \in \mathrm{GL}_d$, define the same hypersurface.

Lemma

A regular set $\mathcal{M}$ is simultaneously self-adjoint if and only if any (and therefore every) corresponding determinantal representation $M(x_0, \dots, x_n)$ is equivalent to some self-adjoint determinantal representation.

Question (2) can be solved by using semidefinite programming.

Assume that $\mathcal{M}$ is simultaneously self-adjoint. Therefore each corresponding determinantal representation is equivalent to some self-adjoint determinantal representation

$$x_0 A_0 + x_1 A_1 + \cdots + x_n A_n, \text{ where all } A_i \in \mathbb{H}^{d \times d}.$$

Matrices admit a positive definite linear combination if and only if

$$\{(x_0, x_1, ..., x_n) \in \mathbb{P}^n \,;\, x_0 A_0 + x_1 A_1 + \cdots + x_n A_n \geq 0\} \neq \emptyset.$$

Semidefinite programming (SDP)

Semidefinite programming is probably the most important new development in optimization in the last 20 years.

The semidefinite programme

is to minimize an affine linear functional l on $\mathbb{R}^n$ subject to a linear matrix inequality (LMI) constraint

$$A_0 + x_1 A_1 + \cdots + x_n A_n \geq 0, \text{ where all } A_i \in \mathbb{H}^{d\times d}.$$

SDP can be efficiently solved:

- theoretically by finding an approximate solution with accuracy ε in a time that is polynomial in $\log(\frac{1}{\varepsilon})$ and in the input size of the problem,
- using interior point methods in many concrete situations.

Which convex sets are feasible sets for SDP?

In other words, given a convex set $\mathcal{C} \subset \mathbb{R}^n$, do there exist matrices such that

$$(*) \quad \mathcal{C} = \{x = (x_1, ..., x_n) \in \mathbb{R}^n \; ; \; A_0 + x_1 A_1 + \cdots + x_n A_n \geq 0\}?$$

We refer to $(*)$ as a linear matrix inequality (LMI) representation of $\mathcal{C}$. Sets having a LMI representation are also called spectrahedra.

Question (2): Given a determinantal representation of a self-adjoint set $\mathcal{M}$, is it also a LMI representation?

In order to describe feasible sets for SDP, we examine the determinant of a LMI representation.

Rigidly convex algebraic interior

Let $q(x) = \det(A_0 + x_1 A_1 + \cdots + x_n A_n)$. Take $x^0 = (x_1^0, \ldots, x_n^0) \in \operatorname{Int} \mathcal{C}$ and normalize the LMI representation by $A_0 + x_1^0 A_1 + \cdots + x_n^0 A_n = \mathrm{Id}$. We restrict the polynomial q to a straight line through x^0: for any $x \in \mathbb{R}^n$ consider

$$q(x^0 + tx) = \det(\mathrm{Id} + t(x_1 A_1 + \cdots + x_n A_n)).$$

Since all the eigenvalues of $x_1 A_1 + \cdots + x_n A_n$ are real, we conclude that $q(x^0 + tx) \in \mathbb{R}[t]$ has only real zeroes. We say that it satisfies the real zero (RZ) condition with respect to $x^0 \in \mathbb{R}^n$. An algebraic interior $\mathcal{C}$ whose minimal defining polynomial satisfies the RZ condition with respect to one and then every point of $\operatorname{Int} \mathcal{C}$ is rigidly convex.

Example

- The circle $\{(x_1, x_2)\ ;\ x_1^2 + x_2^2 \leq 1\}$ is a rigidly convex algebraic interior,
- the "flat TV screen" $\{(x_1, x_2)\ ;\ x_1^4 + x_2^4 \leq 1\}$ is not.

Rigidly convex algebraic interior ↔ LMI

Theorem

Set $\mathcal{C}$ that admits a LMI representation is a rigidly convex algebraic interior. Furthermore, determinant of the LMI representation is a multiple of the minimal defining polynomial of $\mathcal{C}$.

Theorem

A necessary and sufficient condition for $\mathcal{C} \subset \mathbb{R}^2$ to admit a LMI representation is that $\mathcal{C}$ is a rigidly convex algebraic interior. Moreover, the size of the matrices in a LMI representation is equal to the degree a minimal defining polynomial of $\mathcal{C}$.

There can be no exact analogue for $n > 2$.

Lemma

Every pair of 3×3 matrices whose determinant induces a real polynomial is simultaneously self-adjoint.

Kronecker canonical forms for the pencil $x_0 M_0 + x_1 M_1$ can be made self-adjoint by suitable left multiplications:

$$x_1 I + x_2 \begin{pmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{pmatrix} \mapsto x_1 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 0 & a \\ 0 & a & 1 \\ a & 1 & 0 \end{pmatrix},$$

$$x_1 I + x_2 \begin{pmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix} \mapsto x_1 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + x_2 \begin{pmatrix} 0 & a & 0 \\ a & 1 & 0 \\ 0 & 0 & b \end{pmatrix},$$

$$x_1 I + x_2 \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & \overline{b} \end{pmatrix} \mapsto x_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + x_2 \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & \overline{b} \\ 0 & b & 0 \end{pmatrix}.$$

$n = 2$: cubic curve

Pick a basis for $\mathcal{M}$, such that

$$\det(x_0M_0 + x_1M_1 + x_2M_2) = -x_1^2x_2 + x_0^3 + px_0^2x_2 + qx_2^3, \quad p, q \in \mathbb{R}$$

is in the Weierstrass form.
The group action

$$x_0M_0+x_1M_1+x_2M_2 \longrightarrow A(x_0M_0+x_1M_1+x_2M_2)B, \quad A, B \in \mathrm{GL}_3(\mathbb{C})$$

in a unique way reduces the representation to

$$(*) \quad x_0 \,\mathrm{Id} + x_1 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} \frac{t}{2} & l & p + \frac{3}{4}t^2 \\ 0 & -t & -l \\ -1 & 0 & \frac{t}{2} \end{pmatrix}$$

where $t, l \in \mathbb{C}$ satisfy $l^2 = t^3 + pt + q$.

$$(*) \cdot \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ proves:}$$

Proposition

$M = x_0 M_0 + x_1 M_1 + x_2 M_2$ can be in unique way transformed to an equivalent representation

$$\begin{pmatrix} x_2(p + \frac{3}{4}t^2) & x_1 + x_2 l & x_0 + x_2 \frac{t}{2} \\ x_1 - x_2 l & x_0 - x_2 t & 0 \\ x_0 + x_2 \frac{t}{2} & 0 & -x_2 \end{pmatrix}, \text{ where } l^2 = t^3 + pt + q.$$

The set $\{M_0, M_1, M_2\}$ is simultaneously self-adjoint if and only if $t \in \mathbb{R}$ and $l \in i\mathbb{R}$.

Definite triplets

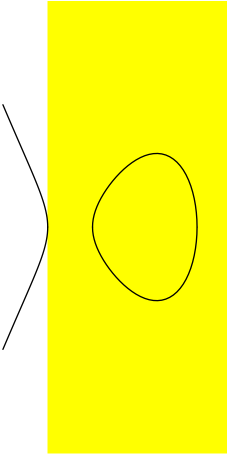
Write $s = i\,l$. Then $(t, s) \subset \mathbb{R}^2$ are points on the affine curve $-s^2 = t^3 + pt + q$.

Theorem

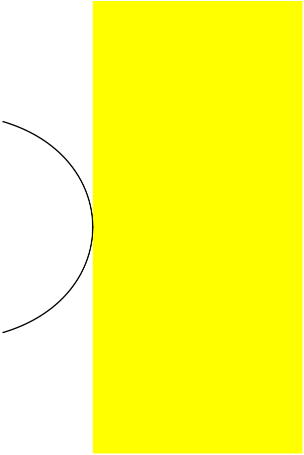
The representation $x_0 A_0 + x_1 A_1 + x_2 A_2$ is definite (LMI representation) if and only if the corresponding point (t, s) lies on the compact component of the affine curve
$-s^2 = t^3 + pt + q$.
A triple of complex hermitean matrices A_0, A_1, A_2 is either definite or A_0, A_1, A_2 have a common self-orthogonal vector.

Smooth cubics $-s^2 = t(t + \theta_1)(t + \theta_2)$

$\theta_1, \theta_1 \in \mathbb{R}$

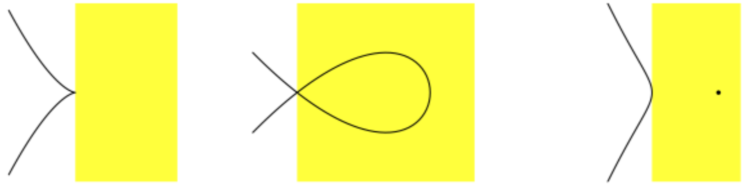


$\theta_1 = \bar{\theta_1}$.



Singular cubics

$$-s^2 = t^3, \quad -s^2 = t^2(t-1), \quad -s^2 = t(t-1)^2.$$



$n = 3$: cubic surface

Proposition

Determinantal representation $M(x_1, x_2, x_3, x_4)$ of a real smooth cubic surface is equivalent to a self-adjoint representation if and only if the double-six corresponding to M, M^t is mutually self-conjugate, i.e.

$$\begin{pmatrix} a_1 & \dots & a_6 \\ b_1 & \dots & b_6 \end{pmatrix}$$

equals to one of the

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ \overline{a_{i_1}} & \overline{a_{i_2}} & \overline{a_{i_3}} & \overline{a_{i_4}} & \overline{a_{i_5}} & \overline{a_{i_6}} \end{pmatrix}.$$

Definite 4-tuples

Let $A(x_0, x_1, x_2, x_3)$ be a self-adjoint determinantal representation of a smooth cubic surface S. The only type of mutually self-conjugate double-six, which does **not** have a self-orthogonal vector is

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ \overline{a_2} & \overline{a_1} & \overline{a_4} & \overline{a_3} & \overline{a_6} & \overline{a_5} \end{pmatrix}.$$

Let $\pi_{11} = \langle a_1, \overline{a_1} \rangle$, $\pi_{22} = \langle a_2, \overline{a_2} \rangle$ be tritangent planes spanned by the lines of S. Then $A(x_0, x_1, x_2, x_3)$ is **definite** if and only if the ovoidal and non-ovoidal piece of S lie in different wedges cut out by π_{11} and π_{22}.

$n \geq 4$

For a set $\mathcal{M}$ with 5 matrices it is enough to check if two of its 4 dimensional subsets are simultaneously self-adjoint.

Theorem

To a 5 dimensional $\mathcal{M}$ we assign a determinantal representation $M = x_0 M_0 + \cdots + x_4 M_4$ which defines a cubic threefold $F(x_0, \ldots, x_4)$ in $\mathbb{P}^4$.
Let π_1 and π_2 be hyperplanes in $\mathbb{P}^4$ such that $F \cap \pi_2$ and $F \cap \pi_2$ are smooth cubic surfaces. Then $\mathcal{M}$ is simultaneously self-adjoint if and only if $M|_{\pi_1}$ and $M|_{\pi_2}$ are equivalent to some self-adjoint representations.

WLG: for $n \geq 4$, we only need to test the sets $\{M_0, M_1, M_2, M_k\}$ for $k = 3, \ldots, n$.

Definite subspaces for $n \geq 4$

To a n dimensional $\mathcal{M}$ we assign a self-adjoint determinantal representation $x_0 A_0 + \cdots + x_n A_n = \left[a_{ij}\right]_{i,j=1}^{3}$, which defines a real cubic hypersurface $F(x_0, \ldots, x_n)$ in $\mathbb{P}^n$.

Proposition

$\mathcal{M}$ is definite if and only if there exist $k_0, \ldots, k_n \in \mathbb{R}$ such that

$$\begin{aligned} L &: \quad a_{11} + a_{22} + a_{33}, \\ Q &: \quad a_{11}a_{22} - a_{12}\overline{a_{12}} + a_{11}aa_{33} - a_{13}\overline{a_{13}} + a_{22}a_{33} - a_{23}\overline{a_{23}}, \\ F &: \quad \det\left[a_{ij}\right] \end{aligned}$$

evaluated in $k_0, \ldots, k_n \in \mathbb{R}$ are all strictly positive.

n-tuples with $n \geq 5$ – 2

Proposition

The representation M is indefinite if and only if the conic $Q = 0$ and its interior $Q > 0$ are entirely included in the $L \cdot F < 0$ part.

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