Spectral variation and geometry of the normal matrices

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Given normal matrices A and B with eigenvalues a_k and b_k we define the spectral distance $\mathrm{sd}(A,B)$ as $\min(\max|a_k - c_k|)$ where c_k runs over all permutations of b_k. The spectral variation inequality $\mathrm{sd}(A,B) \leq \|A - B\|$ (operator norm) holds in several important special cases, but fails in general. In its place we have $\mathrm{sd}(A,B) \leq K\|A-B\|$ where K is a universal constant that is approximately 2.9. However, 2.9 is much larger than any known examples would suggest. A possible way to improve such variation bounds lies in the observation (first made long ago by Rajendra Bhatia) that $\mathrm{sd}(A,B)$ is less than the arc-length of any normal path from A to B. This directs our attention to the problem of finding geodesics (with respect to the operator norm) in the space of normal matrices (and this is also a natural problem independent of applications). Remarkably, to our knowledge no normal geodesics have been identified for sure aside from "short normal paths", ie normal paths from A to B that have the minimum possible length $\|A - B\|$ (although they are not typically straight-line paths). Short normal paths exist, for example, when the eigenvalues of A and B lie on concentric circles (this includes all 2×2 cases). Otherwise the normal geodesic problem appears to be wide open! Even some progress on the 3×3 case would be welcome.