Studying the singularity of LCM-type matrices via semilattice structures and their Möbius functions

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The invertibility of LCM matrices and their Hadamard powers have been studied a lot over the years by many authors. Bourque and Ligh conjectured in 1992 that the LCM matrix $[S] = [[x_i, x_j]]$ on any GCD closed set $S = \{x_1, x_2, \ldots, x_n\}$ is invertible, but in 1997 this was proven false by Haukkanen et al. However, currently there are many open conjectures concerning LCM matrices and their real Hadamard powers presented by Hong. In this presentation we utilize lattice-theoretic structures and the Möbius function to explain the singularity of classical LCM matrices and their Hadamard powers. At the same time we end up disproving some of Hong's conjectures. We apply the mathematics software Sage to show that every 8-element GCD closed set S, for which the LCM matrix [S] is singular, has the same semilattice structure. We also construct a GCD closed set S of odd numbers such that the LCM matrix [S] is singular. Elementary mathematical analysis is applied to prove that for most semilattice structures there exist a set $S = \{x_1, x_2, \ldots, x_n\}$ of positive integers and a real number $\alpha > 0$ such that S possesses this structure and the power LCM matrix $[[x_i, x_j]^\alpha]$ is singular.

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