An elementary proof of Wigner's theorem on quantum mechanical symmetry transformations

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Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{P}_1 = \mathcal{P}_1(\mathcal{H})$ denote the set of *rank-one and self-adjoint projections* on $\mathcal{H}$, i.e.

$$\mathcal{P}_1 = \{\mathbf{P}[\vec{v}] \colon \vec{v} \in \mathcal{H}, \|\vec{v}\| = 1\}$$

where $\mathbf{P}[\vec{v}]$ refers to the projection with precise range $\mathbb{C} \cdot \vec{v}$ (the notations $|\vec{v}\rangle\langle\vec{v}|$ or $\vec{v} \otimes \vec{v}$ are also favourable versions for $\mathbf{P}[\vec{v}]$). We note that the so-called *unit rays* (or *pure states*) of $\mathcal{H}$ and the *one-dimensional subspaces* of $\mathcal{H}$ can be identified with $\mathcal{P}_1$ in a very natural way.

The *transition probability* between two elements $\mathbf{P}[\vec{v}]$ and $\mathbf{P}[\vec{w}]$ is the quantity $\operatorname{tr} \mathbf{P}[\vec{v}]\mathbf{P}[\vec{w}] = |\langle\vec{v}, \vec{w}\rangle|^2$. Wigner's theorem on quantum symmetry transformations is very important in quantum mechanics which belongs to the mathematical foundation of the subject. It states that a mapping on $\mathcal{P}_1$ which preserves the transition probability is induced by a linear or antilinear isometry of the underlying Hilbert space $\mathcal{H}$. We note that there are some other equivalent formulations of the above theorem.

The classical version (i. e. the bijective case) was first stated by E. Wigner in 1931. In his book Wigner himself did not give a rigorous mathematical proof, he said "it is trivial". However, the first such proof (for the bijective case) was given by J. S. Lomont and P. Mendelson, thirty-two years later. One year after that V. Bargmann gave another proof. Several other proofs were given so far to the bijective and non-bijective versions.

In my talk I would like to present a completely new, elementary and very short proof of this famous theorem which is very important in quantum mechanics. We do not assume bijectivity of the mapping or separability of the underlying space like in many other proofs. The advantages of our approach is that it is short, there are no hard calculations, it is very elementary and it works for the general case. As far as we know, such a proof which shares all of these properties was never given before.