Simultaneously self-adjoint sets of matrices

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The article addresses the following two questions:

(1) Consider a set of matrices $\mathcal{M} \subset \mathbb{C}^{d \times d}$. When are all the elements of $\mathcal{M}$ simultaneously equivalent to hermitian matrices under the natural action of $\mathrm{GL}_d(\mathbb{C}) \times \mathrm{GL}_d(\mathbb{C})$? In other words, when do there exist $A, B \in \mathrm{GL}_d(\mathbb{C})$ such that ANB is hermitian for all $N \in \mathcal{M}$?

(2) Assume that the answer to (1) is positive. Is there an element in $\mathcal{M}$ that is equivalent (under this simultaneous equivalence) to a positive definite matrix? In other words, given a set of hermitian $d \times d$ matrices, when do these matrices admit a positive definite linear combination?

We connect three approaches to study the above questions:

- linear algebra (simultaneous linear transformations of a set of matrices to symmetric or hermitian form)
- algebraic geometry (cubic curves, surfaces and hypersurfaces as zero loci of determinants)
- semidefinite programming (linear matrix inequality (LMI) representations)

Computationally both questions are straightforward. Question (1) reduces to a system of linear equations over $\mathbb{R}$. Question (2) is solved by semidefinite programming (at least for small d).

The study of simultaneous classification of matrices is equivalent to the geometric problem of determinantal representations. To the set $\mathcal{M}$ with a basis $\{M_0, \dots, M_n\}$ we assign the matrix $M = x_0 M_0 + x_1 M_1 + \dots + x_n M_n$, a *determinantal representation* of the homogeneous polynomial $\det M$. Definiteness of $\mathcal{M}$ imposes strong constraints on the determinant. For example, LMI hermitian representations induce hyperbolic polynomials / hyperbolicity cones.

For a generic set $\mathcal{M}$ with a chosen basis the corresponding determinant defines a smooth hypersurface of degree d in $\mathbb{P}^n$.

Our main interest is $d = 3$. In this case Question (1) reduces to testing four dimensional subsets in $\mathcal{M}$, where we use the theory of determinantal representations of cubic surfaces in $\mathbb{P}^3$. The reduction uses the theory of determinantal representations of cubic curves in $\mathbb{P}^2$.

We investigate the connection between definiteness and the existence of self-orthogonal vectors.

This is a joint work with T. Košir (University of Ljubljana).

References

[1] A. Buckley, T. Košir, *Determinantal representations of smooth cubic surfaces*, Geometriae Dedicata 125(2007), 115–140.

[2] V. Vinnikov, *LMI Representations of Convex Semialgebraic Sets and Determinantal Representations of Algebraic Hypersurfaces: Past, Present, and Future*, Operator Theory: Advances and Applications 222(2012), 325–349.

[3] V. Vinnikov, *Self-adjoint determinantal representations of real plane curves*, Math. Anal. 296(1993), 453–479.