Spectral theory in max algebra

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The algebraic system max algebra and its isomorphic versions provide an attractive way of describing a class of non-linear problems appearing for instance in manufacturing and transportation scheduling, information technology, discrete event-dynamic systems, combinatorial optimisation, mathematical physics, DNA analysis, ... Max algebra's usefulness arises from a fact that these non-linear problems become linear when described in the max algebra language.

The max algebra consists of the set of non-negative numbers with sum $a \oplus b = \max\{a, b\}$ and the standard product $a \otimes b = ab$, where $a, b \geq 0$. Max algebra is a part of idempotent (tropical) mathematics, since $a \oplus a = a$. The operations between matrices and vectors are defined by analogy with the usual linear algebra. For $A, B \in \mathbb{R}_+^{n \times n}$ and $x \in \mathbb{R}_+^n$ the sum $\oplus$ and the product $\otimes$ in max algebra are defined by $[A \oplus B]_{ij} = \max a_{ij}, b_{ij}$, $[A \otimes B]_{ij} = \max_{k=1,\ldots,n} a_{ik} b_{kj}$ and $[A \otimes x]_i = \max_{j=1,\ldots,n} a_{ij} x_j$. The max eigenproblem is well studied and there are important explicit applications of it in solving the problems mentioned above. In particular, there exists significant analogy with the usual Perron-Frobenius theory. Recently, max algebra techniques have also proved to be very useful in solving classical linear algebra problems (see e.g. [2]). On the other hand, there is little known about the infinite dimensional spectral theory in tropical mathematics, eventhough the need for it has already been explicitly expressed by the tropical society. The main problem here is the definition of the spectrum (at least in the setting of infinite non-negative matrices in max algebra).

References

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