Functions of free noncommuting variables and their differential calculus

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We define a function of d noncommuting variables as a function from a domain in $(\mathbb{C}^{n\times n})^d$ to $\mathbb{C}^{n\times n}$, for all matrix dimensions n, satisfying natural compatibility conditions for different values of n (it has to respect direct sums, and to commute with joint similarity). Our main motivation came from the work of Helton and his coworkers on matrix convexity and matrix positivity, but as it turned out variants of this notion were already considered by J. L. Taylor in his work on functional calculus for noncommuting operators back in early 1970s. The main examples are provided by polynomials and power series in noncommuting variables, and our main result is a kind of noncommutative Taylor series under very weak regularity assumptions (in fact, local boundedness more or less suffices) which shows that in many natural situations this is everything. E.g., given a noncommutative function whose entries are polynomials in matrix entries of the arguments of uniformly bounded degree (with respect to the matrix dimension), this function equals a noncommutative polynomial. To prove these things we construct a kind of noncommutative differential calculus (more precisely, this calculus combines differential calculus and the calculus of finite differences). This should have many applications — e.g. we can establish easily various foundational results on singularities of noncommutative rational functions in terms of their minimal realizations which were previously known only in special situations and required a great ammount of ingenuity and labour to establish.