Spaces of real places

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An $\mathbb{R}$ - place of a formally real field K is a place $\xi : K \longrightarrow \mathbb{R} \cup \{\infty\}$. The set of all $\mathbb{R}$-places of the field K is denoted by $M(K)$. Every $\mathbb{R}$ - place of K is connected with some subset of the space $X(K)$ of orderings of the field K. Namely, if ξ is an $\mathbb{R}$ - place, then there exists an ordering P such that the set

$$A(P) := \{a \in K : \exists q \in \mathbb{Q}^{+}(q \pm a \in P)\}$$

is the valuation ring of ξ. We say that P determines ξ in this case. Any ordering P of the field K determines exactly one $\mathbb{R}$ - place.

The above described correspondence between orderings and $\mathbb{R}$-places defines a surjective map

$$\lambda_K : \mathcal{X}(K) \longrightarrow M(K),$$

which, in turn, allows us to equip $M(K)$ with the quotient topology inherited from $\mathcal{X}(K)$. $M(K)$ is a Hausdorff space. It is also compact as a continuous image of a compact space. But the problem

Which compact, Hausdorff spaces occur as the spaces of real places?

is still open.

We prove that:

- every Boolean space is a space of $\mathbb{R}$ - places of some formally real field K;
- the segment [0,1] is realised as a space of $\mathbb{R}$ - places for some algebraic extension of $\mathbb{R}(X)$;
- the space of $\mathbb{R}$ - places of a function field over any real closure of $\mathbb{R}(X)$ is not a metrizable space and therefore $M(\mathbb{R}(X, Y))$ is not metrizable.