Truncated moment problems: The extremal case

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For a degree $2n$ real d-dimensional multisequence $\beta \equiv \beta^{(2n)} = \{\beta_i\}_{i \in Z_+^d, |i| \leq 2n}$ to have a *representing measure* μ, it is necessary for the associated moment matrix $\mathcal{M}(n)(\beta)$ to be positive semidefinite, and for the algebraic variety associated to β, $\mathcal{V} \equiv \mathcal{V}_\beta$, to satisfy $\operatorname{rank} \mathcal{M}(n) \leq \operatorname{card} \mathcal{V}$ as well as the following *consistency* condition: if a polynomial $p(x) \equiv \sum_{|i| \leq 2n} a_i x^i$ vanishes on $\mathcal{V}$, then $p(\beta) := \sum_{|i| \leq 2n} a_i \beta_i = 0$. In joint work with Lawrence Fialkow and Michael Möller, we employ tools and techniques from algebraic geometry (e.g., Hilbert polynomials, Gröbner and H-bases, representation of positive polynomials) to prove that for the *extremal* case ($\operatorname{rank} \mathcal{M}(n) = \operatorname{card} \mathcal{V}$), positivity of $\mathcal{M}(n)$ and consistency are sufficient for the existence of a (unique, $\operatorname{rank} \mathcal{M}(n)$-atomic) representing measure.

Truncated moment problems (TMP) as above for which the support of a representing measure is required to lie inside a closed set K are called truncated K-moment problems (TKMP). In case K is a semi-algebraic set determined by polynomials $q_1,...,q_m$, the study of TKMP is dual to determining whether a polynomial nonnegative on K belongs to the positive cone consisting of polynomials of degree at most $2n$ which can be expressed as sums of squares, and of squares multiplied by one or more distinct q_i's.

The extremal case, which we have now solved, is inherent in the TMP. A recent result of C. Bayer and J. Teichmann (extending a classical theorem of V. Tchakaloff and its successive generalizations given by I.P. Mysovskikh, M. Putinar, and L. Fialkow and the speaker) implies that if $\beta^{(2n)}$ has a representing measure, then it has a finitely atomic representing measure. In joint work with L. Fialkow, we had previously shown that $\beta^{(2n)}$ has a finitely atomic representing measure if and only if $\mathcal{M}(n) \equiv \mathcal{M}(n)(\beta)$ admits an extension to a positive moment matrix $\mathcal{M}(n+k)$ (for some $k \geq 0$), which in turn admits a rank-preserving (i.e., *flat*) moment matrix extension $\mathcal{M}(n+k+1)$. Further, we proved that any flat extension $\mathcal{M}(n+k+1)$ is an extremal moment matrix for which there is a computable rank $\mathcal{M}(n+k)$-atomic representing measure μ. In this sense, the existence of a representing measure for $\beta^{(2n)}$ is intimately related to the solution of an extremal TMP.