Some illustrated comments on 5×5 golden magic matrices and on 5×5 *Stifelsche Quadrate*

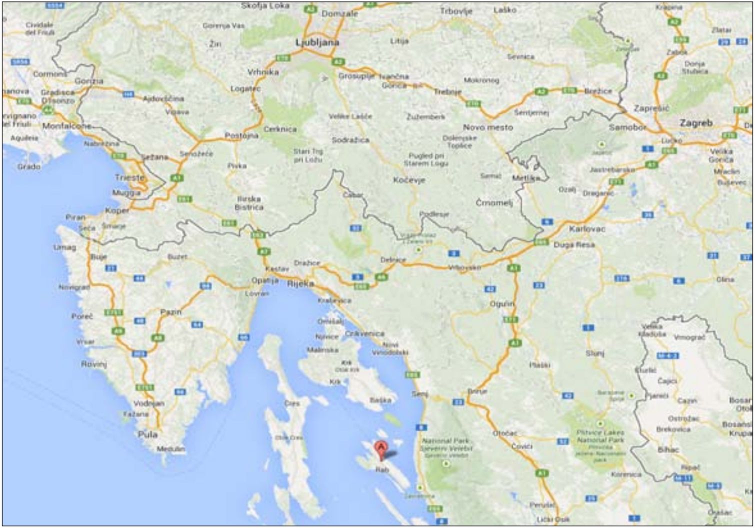
George P. H. Styan[2]

June 10, 2014

[2] To be presented as an invited paper (on Wednesday 11 June 2014) at the 23rd International Workshop on Matrices and Statistics (IWMS-2014), Ljubljana, Slovenia, 8–12 June 2014, and based, in part, on Report 2014-02 from the Department of Mathematics and Statistics, McGill University, Montréal (Québec), Canada. An annotated, indexed & hyperlinked bibliography for magic squares is in active progress. This research, in collaboration with Miguel Angel Amela (General Pico, Argentina), was supported, in part, by the Natural Sciences and Engineering Research Council of Canada (NSERC). This beamer-file was edited on June 10, 2014.

This research on golden magic matrices started with

Cover 1 of *Retour en Dalmatie: le carré magique* (2006), by Jean François Dominis [Giovanni Francesco Dominis/Ivan Franjo Dominis (b. Rab Island, Croatia: 1942)].



Ljubljana (centre top) is the site of the 23rd International Workshop on Matrices and Statistics (IWMS-2014), and Rab Island (centre bottom A) birthplace of Jean François Dominis (b. 1942).

The magic square on Cover 1 of
Retour en Dalmatie: le carré magique
by Jean François Dominis (2006)
may be the most well-known
5×5 magic square and
is at least 450 years old.

It was used by the magician,
occult writer, and astrologer

Heinrich Cornelius Agrippa von Nettesheim (1486–1535)

as a magic-square planetary talisman
for the planet Mars.

Our golden magic matrices $\mathbf{M}$ are 5×5 classic fully-magic matrices where $\mathbf{M}^2$ is a Toeplitz-circulant with top row (p, q, r, r, q) and eigenvalues $a + \Phi b$.

Here $\Phi = (1 + \sqrt{5})/2$ is the Golden Ratio and a, b are rational numbers.

The matrix $\mathbf{M}$ is fully-magic whenever all the rows and columns and the two main diagonals all add up to the same number, the magic sum m.

When the entries are consecutive nonnegative integers then we say that the fully-magic matrix $\mathbf{M}$ is classic.

For example $\mathbf{M} = \mathbf{A}$
the Agrippa–Dominis magic matrix

$$\mathbf{A} = \begin{pmatrix} 11 & 24 & 7 & 20 & 3 \\ 4 & 12 & 25 & 8 & 16 \\ 17 & 5 & 13 & 21 & 9 \\ 10 & 18 & 1 & 14 & 22 \\ 23 & 6 & 19 & 2 & 15 \end{pmatrix}$$

is classic fully-magic with magic sum 65 and golden in that

$$\mathbf{A}^2 = \begin{pmatrix} 605 & 965 & 845 & 845 & 965 \\ 965 & 605 & 965 & 845 & 845 \\ 845 & 965 & 605 & 965 & 845 \\ 845 & 845 & 965 & 605 & 965 \\ 965 & 845 & 845 & 965 & 605 \end{pmatrix}$$

with $p = 605,\ q = 965,\ r = 845.$

We define 5×5 *Stifelsche Quadrate* as
5×5 classic fully-magic matrices
where the 3×3 nucleus
(inner centre submatrix or heart)
is also fully-magic and classic.

For example $\mathbf{M} = \mathbf{S}$,
the Stifel-stamp magic matrix

$$\mathbf{S} = \begin{pmatrix} 3 & 18 & 21 & 22 & 1 \\ 24 & 16 & 11 & 12 & 2 \\ 7 & 9 & 13 & 17 & 19 \\ 6 & 14 & 15 & 10 & 20 \\ 25 & 8 & 5 & 4 & 23 \end{pmatrix}$$

is a *Stifelsche Quadrat* with a
3×3 fully-magic classic nucleus.

Stifelsche Quadrate are so called in
honour of the mathematician and
Augustinian monk

Michael Stifel (1487–1567).

The Stifel-stamp matrix $\mathbf{S} = \{s_{ij}\}$ with magic sum $m = 65$ is not golden but enjoys a plus-flip property

$$(\mathbf{I} + \mathbf{F})\mathbf{S}(\mathbf{I} + \mathbf{F}) = 4m\bar{\mathbf{E}}$$

plus-flip $\Rightarrow s_{33} = \frac{1}{5}m = 13.$

The matrix **S** is not **F**-associated like the Agrippa–Dominis matrix **A**

$$\mathbf{AF} + \mathbf{FA} = \mathbf{A} + \mathbf{FAF} = 2m\bar{\mathbf{E}}$$

which implies the plus-flip property

$$(\mathbf{I} + \mathbf{F})\mathbf{A}(\mathbf{I} + \mathbf{F}) = 4m\bar{\mathbf{E}}.$$

The flip matrix

$$\mathbf{F} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

while the matrix

$$\bar{\mathbf{E}} = \frac{1}{5}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

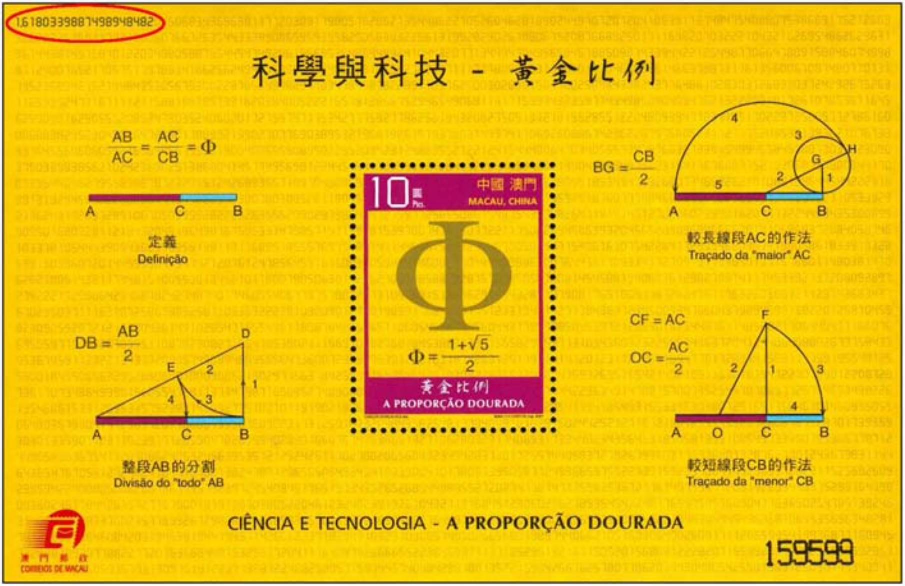
is the 5×5 matrix with every entry equal to $1/5$.

$$\Phi = \tfrac{1}{2}(1+\sqrt{5}) = 2\cos\frac{\pi}{5} = 1 + 2\cos\frac{2\pi}{5} = 1 + \frac{1}{\Phi}$$

$$\simeq 1.6180339887498948482045868343656381177203091\ldots$$

Moreover, the two positive quantities a, b with $a > b$ are said to be in "Golden Ratio" whenever their ratio is the same as the ratio of their sum to the larger of the two quantities, i.e.,

$$\Phi = \frac{a}{b} = \frac{a+b}{a}.$$


1.6180339887498948482
科學與科技 - 黃金比例
AB/AC = AC/CB = Φ
A C B
定義
Definição
10 圓 Ptcs.
中國 澳門
MACAU, CHINA
Φ = (1+√5)/2
黃金比例
A PROPORÇÃO DOURADA
BG = CB/2
較長線段AC的作法
Traçado da "maior" AC
DB = AB/2
整段AB的分割
Divisão do "todo" AB
CF = AC
OC = AC/2
較短線段CB的作法
Traçado da "menor" CB
CIÊNCIA E TECNOLOGIA - A PROPORÇÃO DOURADA
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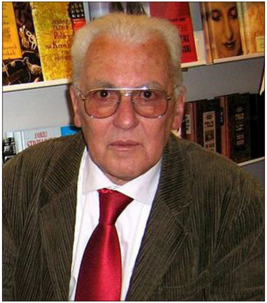
澳門郵政
CORREIOS DE MACAU

As observed by Amela (2014) the
squared Agrippa–Dominis matrix $\mathbf{A}^2$ and its row-flipped partner $\mathbf{FA}^2$

$$\mathbf{A}^2 = \begin{pmatrix} 605 & 965 & 845 & 845 & 965 \\ 965 & 605 & 965 & 845 & 845 \\ 845 & 965 & 605 & 965 & 845 \\ 845 & 845 & 965 & 605 & 965 \\ 965 & 845 & 845 & 965 & 605 \end{pmatrix}, \quad \mathbf{FA}^2 = \begin{pmatrix} 965 & 845 & 845 & 965 & 605 \\ 845 & 845 & 965 & 605 & 965 \\ 845 & 965 & 605 & 965 & 845 \\ 965 & 605 & 965 & 845 & 845 \\ 605 & 965 & 845 & 845 & 965 \end{pmatrix}$$

are, respectively, Toeplitz and Hankel matrices,
named after the mathematicians
Otto Toeplitz (1881–1940) and Hermann Hankel (1839–1873).

According to an obituary article for Hankel, see also *MacTutor* and *Wikipedia*, Hermann Hankel died in Schramberg near Tübingen on 29 August 1873 (age 34). We believe, therefore, that the death-year 1893 on the MaPhyPhil *Marke Individuell* stamp is a typo.



We define the 5×5 Hankel matrix

$$\mathbf{H}_1 = \begin{pmatrix} h_0 & h_1 & h_2 & h_3 & h_4 \\ h_1 & h_2 & h_3 & h_4 & h_5 \\ h_2 & h_3 & h_4 & h_5 & h_6 \\ h_3 & h_4 & h_5 & h_6 & h_7 \\ h_4 & h_5 & h_6 & h_7 & h_8 \end{pmatrix}$$

with the 9 parameters $h_0, h_1, \ldots, h_8$. We may think of the Hankel matrix $\mathbf{H}_1$ as the "one-step backwards circulant without wrap-around".

With wrap-around we get the Hankel circulant $\mathbf{H}_2$ with just the 5 parameters h_0, h_1, h_2, h_3, h_4

$$\mathbf{H}_2 = \begin{pmatrix} h_0 & h_1 & h_2 & h_3 & h_4 \\ h_1 & h_2 & h_3 & h_4 & h_0 \\ h_2 & h_3 & h_4 & h_0 & h_1 \\ h_3 & h_4 & h_0 & h_1 & h_2 \\ h_4 & h_0 & h_1 & h_2 & h_3 \end{pmatrix}.$$

We note that both the Hankel matrix $\mathbf{H}_1$ and the Hankel circulant $\mathbf{H}_2$ are symmetric but that neither is (in general) centrosymmetric.

The square matrix $\mathbf{K}$ is centrosymmetric whenever $\mathbf{FKF} = \mathbf{K}$, where $\mathbf{F}$ is the flip matrix. And bisymmetric whenever it is both centrosymmetric and symmetric.

Moreover, both $\mathbf{H}_1$ and $\mathbf{H}_2$ are "backwards-diagonally constant" in that each descending backwards-diagonal from right to left is constant.

Then the Hankel matrix $\mathbf{H}_1$ equals the Hankel circulant $\mathbf{H}_2$, i.e., $\mathbf{H}_1 = \mathbf{H}_2$, if and only if

$$h_5 = h_0, \quad h_6 = h_1, \quad h_7 = h_2, \quad h_8 = h_3.$$

The Hankel matrix $\mathbf{H}$ is a Hankel circulant if and only if $\mathbf{H}$ is semi-magic, i.e., all its rows and columns add to the magic sum.

We define the 5×5 Toeplitz matrix $\mathbf{T}_1$ as the row–flipped Hankel matrix

$$\mathbf{T}_1 = \mathbf{F}\mathbf{H}_1 = \begin{pmatrix} h_4 & h_5 & h_6 & h_7 & h_8 \\ h_3 & h_4 & h_5 & h_6 & h_7 \\ h_2 & h_3 & h_4 & h_5 & h_6 \\ h_1 & h_2 & h_3 & h_4 & h_5 \\ h_0 & h_1 & h_2 & h_3 & h_4 \end{pmatrix}$$

and the Toeplitz-circulant $\mathbf{T}_2$ as the row-flipped Hankel-circulant $\mathbf{H}_2$ with just the 5 parameters h_0, h_1, h_2, h_3, h_4

$$\mathbf{T}_2 = \mathbf{F}\mathbf{H}_2 = \begin{pmatrix} h_4 & h_0 & h_1 & h_2 & h_3 \\ h_3 & h_4 & h_0 & h_1 & h_2 \\ h_2 & h_3 & h_4 & h_0 & h_1 \\ h_1 & h_2 & h_3 & h_4 & h_0 \\ h_0 & h_1 & h_2 & h_3 & h_4 \end{pmatrix}.$$

We note that neither the Toeplitz matrix $\mathbf{T}_1$ nor the Toeplitz-circulant $\mathbf{T}_2$ is (in general) symmetric or (in general) centrosymmetric.

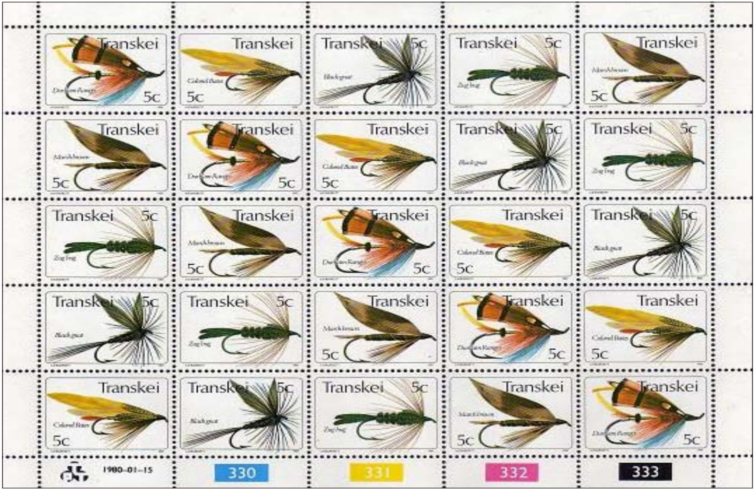
Moreover, both $\mathbf{T}_1$ and $\mathbf{T}_2$ are "forwards-diagonally constant" in that each descending forwards-diagonal from left to right is constant.

Then the Toeplitz matrix $\mathbf{T}_1$ equals the Toeplitz circulant $\mathbf{T}_2$ if and only if

$$h_5 = h_0, \quad h_6 = h_1, \quad h_7 = h_2, \quad h_8 = h_3.$$

The Toeplitz matrix $\mathbf{T}$ is a Toeplitz circulant if and only if $\mathbf{T}$ is semi-magic.

Loly & Styan (2010) discussed 5×5 philatelic Latin squares in some detail. Here we present a 5×5 philatelic Latin square from Transkei which is also a "philatelic Toeplitz circulant".



Following the seminal book *Circulant Matrices*, by Philip J. Davis (1979/1994), we define the 5×5 Fourier matrix

$$\mathbf{G} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 \\ 1 & \omega^2 & \omega^4 & \omega & \omega^3 \\ 1 & \omega^3 & \omega & \omega^4 & \omega^2 \\ 1 & \omega^4 & \omega^3 & \omega^2 & \omega \end{pmatrix}$$

where $\omega = \omega_1$ is the first primitive 5th root of unity, and with the imaginary unit $i = \sqrt{-1}$

$$\omega_1 = \omega = \exp(2\pi i/5) = \cos(2\pi/5) + i \sin(2\pi/5) = \tfrac{1}{4}\Big(-1 + \sqrt{5} + i\sqrt{10 + 2\sqrt{5}}\Big)$$

$$\omega_2 = \omega^2 = \exp(4\pi i/5) = \cos(4\pi/5) + i \sin(4\pi/5) = \tfrac{1}{4}\Big(-1 - \sqrt{5} + i\sqrt{10 + 2\sqrt{5}}\Big)$$

$$\omega_3 = \omega^3 = \exp(6\pi i/5) = \cos(6\pi/5) + i \sin(6\pi/5) = \tfrac{1}{4}\Big(-1 - \sqrt{5} - i\sqrt{10 + 2\sqrt{5}}\Big)$$

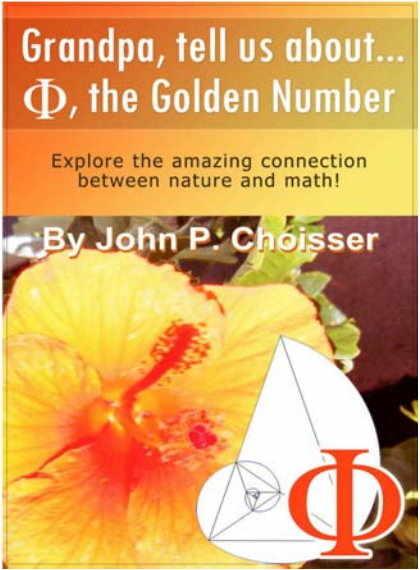
$$\omega_4 = \omega^4 = \exp(8\pi i/5) = \cos(8\pi/5) + i \sin(8\pi/5) = \tfrac{1}{4}\Big(-1 + \sqrt{5} - i\sqrt{10 + 2\sqrt{5}}\Big).$$

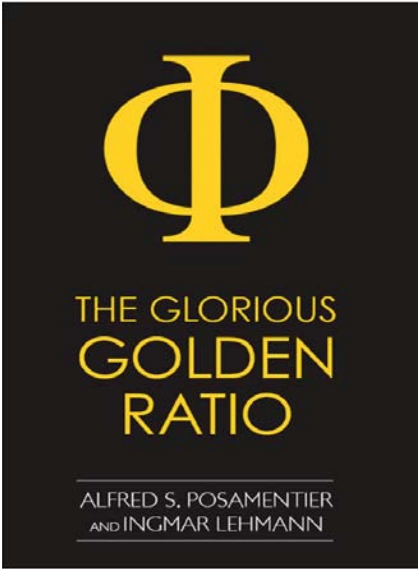
It follows at once that the roots of unity are golden in that

$$\omega_1 + \omega_4 = \omega + \omega^4 = \Phi - 1$$

$$\omega_2 + \omega_3 = \omega^2 + \omega^3 = -\Phi.$$

Piero della Francesca (1415–1492):s *de Divina Proportione*


Grandpa, tell us about...
Φ, the Golden Number
Explore the amazing connection between nature and math!
By John P. Choisser
Φ


Φ
THE GLORIOUS
GOLDEN
RATIO
ALFRED S. POSAMENTIER
AND INGMAR LEHMANN

According to Gilbert Strang in the Fourth Edition of his
Introduction to Linear Algebra (2009):

"The Fourier matrix is absolutely the most important complex matrix in mathematics and science and engineering."

The Fourier matrix is named after the well-known French mathematician and physicist Jean Baptiste Joseph Fourier (1768–1830) best known for Fourier analysis and the Fast Fourier transform (FFT), and for initiating the investigation of Fourier series and their applications to problems of heat transfer and vibrations.

Fourier transform spectroscopy is a measurement technique whereby spectra are collected based on measurements of the coherence of a radiative source, using time-domain or space-domain measurements of the electromagnetic radiation or other type of radiation.

We find that

$$\mathbf{GFG} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \omega^4 & 0 & 0 & 0 \\ 0 & 0 & \omega^3 & 0 & 0 \\ 0 & 0 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & 0 & \omega \end{pmatrix}$$

and so the Fourier matrix **G** diagonalizes the flip matrix **F** and the eigenvalues of both **GFG** and $\mathbf{FG}^2$ are

$$1, \omega, \omega^2, \omega^3, \omega^4.$$

We find this result on diagonalizing the flip matrix **F** to be particularly interesting (and surprising).

The flip matrix **F** is, however, a Hankel circulant.


J.B. JOSEPH
FOURIER
1768-1830
$F(x) = a_0 + \sum_{k=1}^{\infty} a_k \sin kx + b_k \cos kx$
0.55
LIECHTENSTEIN
SCHE POST AG


Venera 15 und 16
Fourier-Spektrometer
DDR
DDR 70
25 JAHRE BEMANNTER WELTRAUMFLUG
1986

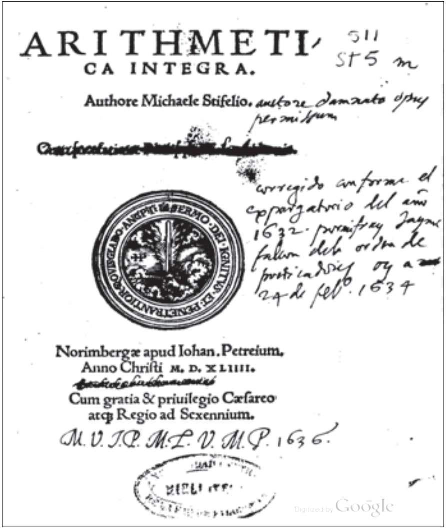
In the 5×5 Fourier matrix $\mathbf{G}$ we define the 4×4 Fourier-powers magic matrix $\mathbf{G}_{\mathrm{p}}$ of the powers of ω in the lower-right 4×4 submatrix of $\sqrt{5}\mathbf{G}$

$$\mathbf{G} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 \\ 1 & \omega^2 & \omega^4 & \omega & \omega^3 \\ 1 & \omega^3 & \omega & \omega^4 & \omega^2 \\ 1 & \omega^4 & \omega^3 & \omega^2 & \omega \end{pmatrix}, \qquad \mathbf{G}_{\mathrm{p}} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \\ 3 & 1 & 4 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix}.$$

- $\mathbf{G}_{\mathrm{p}}$ is bisymmetric fully-magic Latin with rank 3, index 1.
- $\mathbf{G}_{\mathrm{p}}$ is EP, i.e., $\mathbf{G}_{\mathrm{p}}\mathbf{G}_{\mathrm{p}}^{+} = \mathbf{G}_{\mathrm{p}}^{+}\mathbf{G}_{\mathrm{p}}$
- $\mathbf{G}_{\mathrm{p}}$ is $\mathbf{V}$-associated with centrosymmetric involutory $\mathbf{V}$ and so $\mathbf{G}_{\mathrm{p}}^{+}$ is fully-magic.

16	81	79	77	75	11	13	15	2
78	28	65	63	61	25	27	18	4
76	62	36	53	51	35	30	20	6
74	60	50	40	45	38	32	22	8
9	23	33	39	41	43	49	59	73
10	24	34	44	37	42	48	58	72
12	26	52	29	31	47	46	56	70
14	64	17	19	21	57	55	54	68
80	1	3	5	7	71	69	67	66

The first bordered magic squares that we have found published in Europe are by Michael Stifel (1487–1567) in his *Arithmetica Integra* (1544). The inner 5×5 fully-magic nucleus in the 9×9 fully-magic square, left panel above, is flagged with a red box. The 5×5 bordered magic square on the stamp is, however, different.



ARITHMETICA INTEGRA. 511 St5 m

Authore Michaele Stifelio. auctore damnato c[on] s[u]u permissum

[redacted]

corregido conforme el expurgatorio del año 1632. por mi fray Jayme Fallon de la orden de predicadores a 24 de feb. 1634

Norimbergæ apud Iohan. Petreium.
Anno Christi M. D. XLIIII.

[redacted]

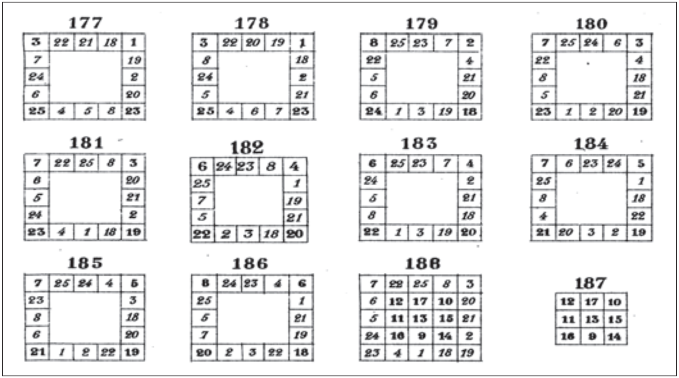
Cum gratia & priuilegio Cæsareo atq; Regio ad Sexennium.

M. V. I. P. M. L. V. M. P. 1636.

Digitized by Google

Bernard Violle (fl. 1815–1838) in his extensive *Traité Complet des Carrés Magiques* [vol. 1, pp. 46–85 (1837)] identified 594 bordered 5×5 classic fully-magic squares, including 10 [$\times\ 3! \times 3! \times 8 = 2880$] *Stifelsche Quadrate* with a consecutively-numbered 3×3 nucleus.

The borders of these 10 magic squares were illustrated by Mikhail Frolov (1884):



Even with the internet we have found it challenging to find out how many bordered 5×5 classic fully-magic squares there are in all and who first discovered the counts and when.

The first published count seems to be in 1837 with the detailed listing of 594 by Bernard Violle (fl. 1815–1838) in his extensive *Traité Complet des Carrés Magiques* [vol. 1, pp. 46–85].

In April 1902, Charles Planck (1856–1935), in a one-page letter to the editor in *Nature*, identified 602 and so 8 more than the 594 found by Violle.

Planck, like Violle, observed that these magic squares may be classified into 26 types and Planck gave the frequencies per type.

A month later in May 1902, John Willis (b. 25 February 1819), also in a letter to the editor in *Nature*, identified 603, and so one more than Planck, and presented a complete list of 21 bordered magic squares of Planck's Type R with smallest entries $5, 7$ in the 3×3 nucleus.

Willis in his 1909 book *Easy Methods of Constructing the Various Types of Magic Squares* [pp. 97–102] listed in detail 604, and so 2 more than Planck and 1 more than in his [Willis] May 1902 *Nature* letter.

Henry Ernest Dudeney (1857–1930), however, had announced already 3 years earlier in 1906 that there are 605 $[= 174240/288]$ bordered 5×5 classic fully-magic squares in all.

We believe that this total of 605 is it!

Digitised newspapers and more

Showing: 1 - 5 of **5** Refine search Sort by: Relevance Sort

THE WORLD OF PUZZLEDOM. (Copyright.) UNSOLVED PROBLEMS.
The Queenslander (Brisbane, Qld. : 1866 - 1939) **Saturday 27 July 1907** p 6 Article

... THE WORLD OF PUZZLEDOM. (Copyright.) UNSOLVED PROBLEMS. By HENRY E. **DUDENEY,** Author of "The Canterbury Puzzles," "The Adventures of the Puzzle Club," Etc., Etc. Recently we described a certain ... exactly **174,240** such squares may be formed of one par ticular kind only -the bordered square, in which ... 874 words

THE WORLD OF PUZZLEDOM. (Copyright.) UNSOLVED PROBLEMS.
Western Mail (Perth, WA : 1885 - 1954) **Saturday 29 February 1908** p 42 Article

... THE WORLD OF PROBLEMS. (By Henry E. **Dudeney.** Author of "The Canterbury Puzzles," The Adventures of the Puzzle Club," etc., etc. In UNSOLVED PROBLEMS. _ - lu a recent issue we described a certain ... **174,240** such squares may be formed of one particular kind only-the bordered square, in which the ... 868 words

THE WORLDS GREATEST PUZZLES.
Chronicle (Adelaide, SA : 1895 - 1954) **Saturday 8 September 1906** p 31 Article Illustrated

... demon strated to be quite impossible of solution (writes H. T. **Dudeney,** known as 'The Puzzle King')- Let ... before we can hope to solve the puzzle. But it is startling to find that exactly **174,240** such squares ... 1112 words

THE WORLD'S GREATEST PUZZLES.
Snowy River Mail (Orbost, Vic. : 1914 - 1918) **Friday 5 October 1917 Edition: WEEKLY** p 3 Article

... THE WORLD'S GREATEST PUZZLES. By 1y1. Henry T. **Dudeney,** "The Puzzle S King." Probing into the secrets of. Nature is a passion with all men; only we select different lines of research. Men have ... **174,240".**. such squares may. be' fornmed .of one par ticular-" : kind "only-the' "bordered suunre, in which ... 1126 words

THE WORLD'S GREATEST PUZZLES.
The Gundagai Independent and Pastoral, Agricultural and Mining Advocate (NSW : 1898 - 1928) **Wednesday 21 August 1907** p 4 Article

... THE WORLD'S GREATEST PUZZLES. ? » ? By Henry T. **Dudeney,** 'The Puzzle King." King.' / ? f ? = Probing into the secrets of Nature is a passion with all men ; only we select different lines of ... before we can hope to solve the puzzle. But it is startling to find that exactly **174,240** such squares ... 1085 words

Henry Ernest Dudeney (1857–1930) in his Introduction to *The Canterbury Puzzles and Other Curious Problems* (1907/1908) wrote

If we wish to make a magic square of the 16 numbers, 1 to 16, there are just 880 different ways of doing it, not counting reversals and reflections. This has been finally proved of recent years.*

*But how many magic squares may be formed with the 25 numbers, 1 to 25, nobody knows**. We shall have to extend our knowledge in certain directions before we can hope to solve this puzzle.*

But it is surprising to find that exactly 174240 such squares may be formed of one particular restricted kind only — the bordered square, in which the nucleus of 9 cells is itself magic.

And I have shown how this number may be at once doubled by merely converting every bordered square — by a simple rule — into a non-bordered one.

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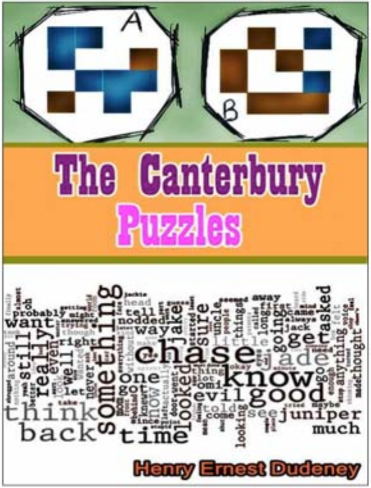
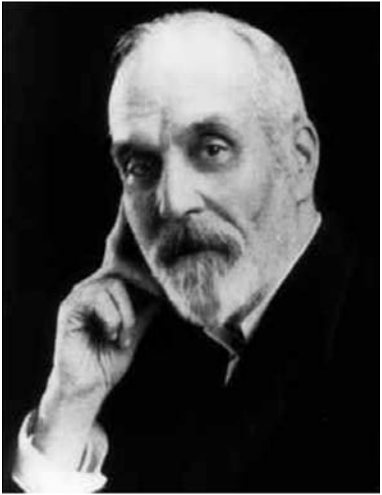
*We believe the first proof that there are precisely 880 classic 4×4 magic squares was by Friedrich Fitting in 1931.

**Thanks to Richard Schroeppel in 1973, we now know, however, that there are

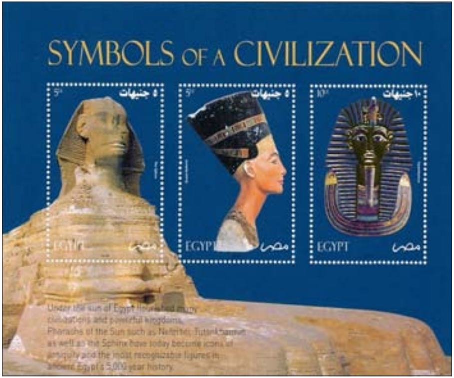
$$275,305,224 \simeq 275 \times 10^6.$$


A
B
The Canterbury
Puzzles
Henry Ernest Dudeney



The "Dudeney total" = $174240 = 605 \times 288$

We conjecture that Dudeney announced this total of 174240 before 8 September 1906, possibly in *The Weekly Dispatch*, a British newspaper in which he published extensively, sometimes under the pseudonym Sphinx.



This Dudeney total = 174240 was confirmed by [Major] John Chaplyn Burnett (1863–1943) in his 1936 book *Easy Methods for the Construction of Magic Squares.*

Burnett acknowledged receiving "many valuable hints" from Planck.

The Dudeney total = 174240 was also confirmed very recently by Harry White (8 April 2014).

Harry White observed that for order 6, the number of distinct concentric magic squares is $736,347,893,760 \simeq 7.36 \times 10^{11}$.

$174240 = 10 \times 11^2 \times 12^2$ sq. feet = 4 acres.

It seems that all 174240 classic bordered magic matrices satisfy the plus-flip property

$$(\mathbf{I}+\mathbf{F})\mathbf{M}(\mathbf{I}+\mathbf{F}) = \mathbf{M}+\mathbf{FMF}+(\mathbf{MF}+\mathbf{FM}) = 4m\bar{\mathbf{E}}. \quad (1)$$

In 2007 we considered classic fully-magic matrices which satisfied the minus-flip property

$$(\mathbf{I}-\mathbf{F})\mathbf{M}(\mathbf{I}-\mathbf{F}) = \mathbf{M}+\mathbf{FMF}-(\mathbf{MF}+\mathbf{FM}) = \mathbf{0}. \quad (2)$$

We then called $\mathbf{MF}+\mathbf{FM}$ the "singly-smoothed" partner to $\mathbf{M}$ and $\mathbf{M}+\mathbf{FMF}$ its "doubly-smoothed" partner.

We then said that the matrix $\mathbf{M}$ is smooth whenever its singly- and doubly-smoothed partners coincide and so a smooth $\mathbf{M}$ satisfies the minus-flip property.

Trump (2007) identified 1004 smooth classic not $\mathbf{F}$-associated 5×5 fully-magic matrices with the minus-flip property, e.g., the smoothie

$$\mathbf{U} = \begin{pmatrix} 25 & 2 & 20 & 11 & 7 \\ 3 & 21 & 17 & 10 & 14 \\ 8 & 15 & 1 & 19 & 22 \\ 5 & 23 & 9 & 12 & 16 \\ 24 & 4 & 18 & 13 & 6 \end{pmatrix}$$

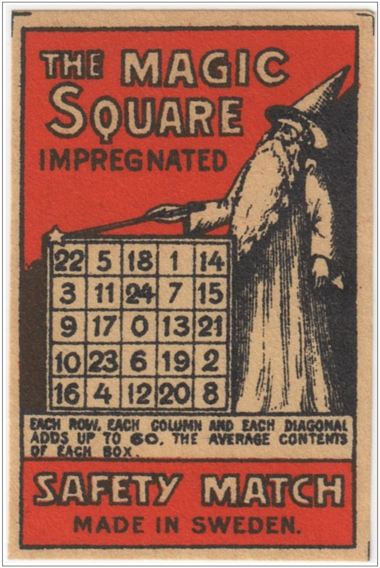
has the minus-flip property but is not golden:

$$\mathbf{U}^2 = \begin{pmatrix} 1014 & 673 & 779 & 898 & 861 \\ 660 & 988 & 776 & 868 & 933 \\ 876 & 871 & 983 & 771 & 724 \\ 710 & 968 & 896 & 808 & 843 \\ 965 & 725 & 791 & 880 & 864 \end{pmatrix}$$

is not a Toeplitz circulant. We recall that the magic matrix $\mathbf{M}$ is $\mathbf{F}$-associated whenever

$$\mathbf{M}+\mathbf{FMF} = \mathbf{MF}+\mathbf{FM} = 2m\bar{\mathbf{E}}.$$

$\mathbf{F}$-associated $\Leftrightarrow$ plus-flip (1) $\cap$ minus-flip (2).



We first found this
Matchbox Label Sweden with Magic Square,
on the "Pinterest profile" on Magic Squares by
Maria João Lagarto.

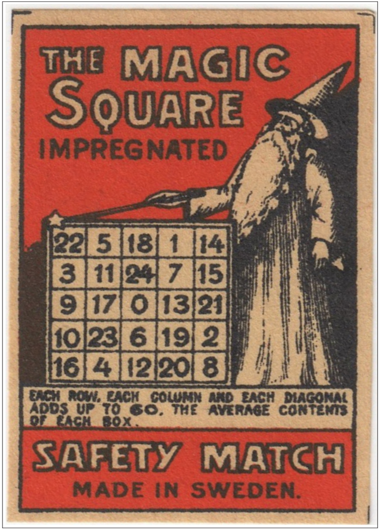
The matchbox-wizard matrix

$$\mathbf{X} = \begin{pmatrix} 22 & 5 & 18 & 1 & 14 \\ 3 & 11 & 24 & 7 & 15 \\ 9 & 17 & 0 & 13 & 21 \\ 10 & 23 & 6 & 19 & 2 \\ 16 & 4 & 12 & 20 & 8 \end{pmatrix}$$

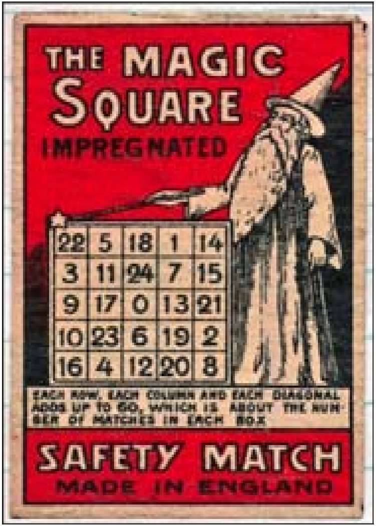
has magic sum $m = 60$
the average contents of each box.

The matrix $\mathbf{X}$ is nonsingular and its inverse $\mathbf{X}^{-1}$ is fully-magic. Both $\mathbf{X}$ and $\mathbf{X}^{-1}$ are pandiagonal and $\mathbf{Q}$-associated.

Also made in England? When???



"... adds up to 60,
the average contents of each box."



" ... adds up to 60,
which is about the number of matches in each box."

The matchbox magic matrix

$$\mathbf{X} = \begin{pmatrix} 22 & 5 & 18 & 1 & 14 \\ 3 & 11 & 24 & 7 & 15 \\ 9 & 17 & 0 & 13 & 21 \\ 10 & 23 & 6 & 19 & 2 \\ 16 & 4 & 12 & 20 & 8 \end{pmatrix}$$

is nonsingular and its inverse

$$\mathbf{X}^{-1} = \frac{1}{600}\begin{pmatrix} 27 & -23 & -3 & 7 & 2 \\ 7 & 2 & 2 & 27 & -28 \\ 2 & 22 & -18 & 2 & 2 \\ -23 & 2 & 2 & -3 & 32 \\ -3 & 7 & 27 & -23 & 2 \end{pmatrix}$$

is fully-magic.

Both $\mathbf{X}$ and $\mathbf{X}^{-1}$ are pandiagonal and golden $\mathbf{Q}$-associated:

$$\mathbf{QX} + \mathbf{XQ} = \mathbf{0} = \mathbf{QX}^{-1} + \mathbf{X}^{-1}\mathbf{Q}$$

where the golden fully-magic Toeplitz circulant

$$\mathbf{Q} = \begin{pmatrix} 0 & 1 & -1 & -1 & 1 \\ 1 & 0 & 1 & -1 & -1 \\ -1 & 1 & 0 & 1 & -1 \\ -1 & -1 & 1 & 0 & 1 \\ 1 & -1 & -1 & 1 & 0 \end{pmatrix}$$

which has golden non-magic eigenvalues

$$\pm\sqrt{5} = \pm(2\Phi - 1)$$

each twice. Moreover

$$\tfrac{1}{5}\mathbf{Q}^2 = \mathbf{I} - \bar{\mathbf{E}} = \mathbf{C}$$

the centering matrix.

I was very pleased to meet Richard Schroeppel at the recent 500th Anniversary Celebration of Albrecht Dürer's *Melencolia I* in New York City (17–18 May 2014).

There I asked him about the count for 6×6 classic fully-magic matrices, which is not known but is estimated in Trump's Table to be

$$(1.775399 \pm 0.000042) \times 10^{19}$$

with probability 99%.

In his talk at the Dürer celebration, Schroeppel presented a magic pattern which holds for any 5×5 fully-magic matrix $\mathbf{M} = \{m_{ij}\}$ with magic sum m.

We define the Schroeppel–pattern matrix

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -3 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

Then Schroeppel's magic pattern says the sum of all the elements in the Hadamard product

$$\mathbf{P} \odot \mathbf{M} = \begin{pmatrix} 0 & m_{12} & 0 & m_{14} & 0 \\ m_{21} & 0 & 0 & 0 & m_{25} \\ 0 & 0 & -3m_{33} & 0 & 0 \\ m_{41} & 0 & 0 & 0 & m_{45} \\ 0 & m_{52} & 0 & m_{54} & 0 \end{pmatrix}$$

is equal to the magic sum m.

Let

$$\mathbf{b} = \mathrm{vec}\mathbf{P}$$

and let $\mathbf{e}_h$ denote the $h \times 1$ column vector with every entry equal to 1. Then we write Schroeppel's magic pattern as

$$\mathbf{e}_5'(\mathbf{P} \odot \mathbf{M})\mathbf{e}_5 = m_{12} + m_{14} + m_{21} + m_{25} - 3m_{33} + m_{41} + m_{45} + m_{52} + m_{54} = m$$

or as

$$\mathbf{e}_5'(\mathbf{P} \odot \mathbf{M})\mathbf{e}_5 = (\mathrm{vec}\mathbf{P})'\mathrm{vec}\mathbf{M} = \mathbf{b}'\mathrm{vec}\mathbf{M} = m. \tag{3}$$

We define a 25×12 incidence matrix $\mathbf{A}$ such that $\mathbf{M}$ is fully-magic if and only if

$$\mathbf{A}'\mathrm{vec}\mathbf{M} = m\,\mathbf{e}_{12}. \tag{4}$$

Then we find that the 12×1 vector $\mathbf{x} = \mathbf{A}^+\mathbf{b}$ is such that $\mathbf{b} = \mathbf{A}\mathbf{x}$ and $\mathbf{x}'\mathbf{e}_{12} = 1$.

Premultiplying (4) by $\mathbf{x}'$ yields (3). ■

$$
\mathbf{A} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0
\end{pmatrix}, \qquad
\mathbf{x} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ 1 \\ 1 \\ -2 \\ -2 \end{pmatrix}, \qquad
\mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ -3 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}.
$$

Motivated by Schroeppel's magic pattern we define the bisymmetric 4-parameter pattern matrix $\hat{\mathbf{P}} =$

$$\frac{1}{t}\begin{pmatrix} 2a+d & a+b & a+c & a+b & 2a+d \\ a+b & 2b+d & b+c & 2b+d & a+b \\ a+c & b+c & 2c+2d & b+c & a+c \\ a+b & 2b+d & b+c & 2b+d & a+b \\ 2a+d & a+b & a+c & a+b & 2a+d \end{pmatrix}$$

where

$$t = 2(2a+2b+c+d)$$

$$\mathrm{vec}\hat{\mathbf{P}} = \mathbf{A}\hat{\mathbf{x}}; \qquad \hat{\mathbf{x}} = \mathbf{A}^{+}\mathrm{vec}\hat{\mathbf{P}}$$

$$\hat{\mathbf{x}} = \frac{1}{t}(a, b, c, b, a, a, b, c, b, a, d, d)'$$

with

$$\hat{\mathbf{x}}'\mathbf{e}_{12} = 1.$$

For Schroeppel's magic pattern $a = b$ and

$$a = \tfrac{1}{2},\ b = \tfrac{1}{2};\ c = -\tfrac{1}{2},\ d = -1; \qquad t = 1$$

and the Schroeppel–pattern matrix

$$\mathbf{P}_1 = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -3 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

For example, with Agrippa–Dominis

$$24 + 20 + 4 + 16 - 39 + 10 + 22 + 6 + 2 = 65.$$

If, however, we choose $b = c$ and

$$a = \tfrac{1}{2},\ b = -\tfrac{1}{2};\ c = -\tfrac{1}{2},\ d = 1$$

then

$$t = 2(2a + 2b + c + d) = 1$$

and the pattern matrix

$$\mathbf{P}_2 = \begin{pmatrix} 2 & 0 & 0 & 0 & 2 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 2 \end{pmatrix}$$

It follows that

$$2m_{11} + 2m_{15} - m_{23} - m_{32} + m_{33} - m_{34}$$
$$- m_{43} + 2m_{51} + 2m_{55} = m.$$

For example, with Agrippa–Dominis

$22 + 6 - 25 - 5 + 13 - 21 - 1 + 46 + 30 = 65.$

On the other hand, if we choose $a = c$ and

$$a = \tfrac{1}{2},\ b = -\tfrac{1}{2}\ c = \tfrac{1}{2},\ d = 1$$

then

$$t = 2(2a + 2b + c + d) = 3$$

and the pattern matrix

$$\mathbf{P}_3 = \frac{1}{3}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 2 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

And so we see that for any 5×5 fully-magic matrix $\mathbf{M} = \{m_{ij}\}$ its 3×3 nucleus

$$\mathbf{M}_{[3\times3]} = \begin{pmatrix} m_{22} & m_{23} & m_{24} \\ m_{32} & m_{33} & m_{34} \\ m_{42} & m_{43} & m_{44} \end{pmatrix}$$

is such that

$$\frac{1}{3}\big(2m_{22} + m_{23} + 2m_{24} + m_{32} + 3m_{33} + m_{34} + 2m_{42} + m_{43} + 2m_{44}\big) = m.$$

For example, with Agrippa–Dominis

$$\frac{1}{3}\big(24+25+16+5+39+21+36+1+28\big) = \frac{195}{3} = 65.$$

The pattern matrices $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3$

Three pattern matrices for 5×5 fully-magic matrices:

$$\mathbf{P_1} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -3 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

$$\mathbf{P_2} = \begin{pmatrix} 2 & 0 & 0 & 0 & 2 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$$\mathbf{P_3} = \frac{1}{3} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 2 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Each pattern matrix

1. is bisymmetric (centrosymmetric and symmetric)
2. has 16 entries equal to 0 and 9 nonzero entries.

For Agrippa–Dominis

$\mathbf{P_1}$: $24 + 20 + 4 + 16 - 39 + 10 + 22 + 6 + 2 = 65$

$\mathbf{P_2}$: $22 + 6 - 25 - 5 + 13 - 21 - 1 + 46 + 30 = 65$

$\mathbf{P_3}$: $\frac{1}{3}\big(24+25+16+5+39+21+36+1+28\big) = \frac{195}{3} = 65.$

Any others? $\mathbf{P}_4, \ldots$

The pattern matrix $\mathbf{P}_4$

We seek a pattern matrix $\mathbf{P}_4$ that

1. is bisymmetric (centrosymmetric and symmetric)
2. has 16 entries equal to 0 and 9 nonzero entries

but $\neq \mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3$ above or

$$\mathbf{P}_{\rm dgs} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix} = \frac{1}{2}(\mathbf{I} + \mathbf{F})$$

$$\mathbf{P}_{\rm rc} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

We recall the Agrippa–Dominis $\mathbf{A}$, Stifel-stamp $\mathbf{S}$, Matchbox-wizard $\mathbf{X}$