ON A NEW FAMILY OF WEIGHTED TLS ALGORITHMS FOR EIV-MODELS WITH ARBITRARY DISPERSION MATRICES

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The EIV Model with nonsingular covariance matrices

- The EIV-Model

$$y = \underbrace{(A - E_A)}_{n\times m}\xi + e_y, \quad \operatorname{rk} A = m < n,$$

$$e := \begin{bmatrix} e_y \\ e_A := \operatorname{vec} E_A \end{bmatrix} \sim \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \sigma_0^2 \begin{bmatrix} \underset{n\times n}{P_y^{-1}} & 0 \\ 0 & \underset{nm\times nm}{P_A^{-1}} \end{bmatrix} =: \sigma_0^2 Q = \sigma_0^2 P^{-1} \right)$$

- where:

y	is the $n \times 1$ observation vector;
A	is the $n \times m$ (random) coefficient matrix with full column rank;
E_A	is the $n \times m$ (unknown) random error matrix associated with A;
ξ	is the $m \times 1$ (unknown) parameter vector;
e_y	is the $n \times 1$ (unknown) random error vector associated with y;
e_A	is the $nm \times 1$ vectorial form of the matrix E_A;
σ_0^2	is the (unknown) variance component;
Q	is the $n(m+1) \times n(m+1)$ block-diagonal pos.-def. cofactor matrix;
$P := Q^{-1}$	is the corresponding block-diagonal pos.-def. weight matrix.

- The model generalizes the one used by Schaffrin and Wieser (2008) where a Kronecker product structure for $Q_A = P_A^{-1} = Q_0 \otimes Q_x$ was assumed.

The weighted Total Least-Squares Solution

- Lagrange's target function (with λ as an $n \times 1$ vector of Lagrange multipliers):

$$\Phi(e_y, e_A, \xi, \lambda) := e_y^T P_y e_y + e_A^T P_A e_A + 2\lambda^T [y - A\xi - e_y + (\xi^T \otimes I_n) e_A] = \text{stationary}$$

- Euler-Lagrange necessary conditions:

$$\frac{1}{2}\frac{\partial \Phi}{\partial e_y} = P_y \tilde{e}_y - \hat{\lambda} \doteq 0 \qquad (1) \Rightarrow \tilde{e}_y = Q_y \hat{\lambda} \qquad (1')$$

$$\frac{1}{2}\frac{\partial \Phi}{\partial e_A} = P_A \tilde{e}_A + (\hat{\xi} \otimes I_n)\hat{\lambda} \doteq 0 \qquad (2) \Rightarrow \tilde{e}_A = -Q_A (\hat{\xi} \otimes I_n)\hat{\lambda} \qquad (2')$$

$$\frac{1}{2}\frac{\partial \Phi}{\partial \xi} = -(A - \tilde{E}_A)^T \hat{\lambda} \doteq 0 \qquad (3) \Rightarrow A^T \hat{\lambda} = (\hat{\lambda}^T \otimes I_m)\,\mathrm{vec}(\tilde{E}_A^T) \qquad (3')$$

$$\frac{1}{2}\frac{\partial \Phi}{\partial \lambda} = y - A\hat{\xi} - \tilde{e}_y + (\hat{\xi}^T \otimes I_n)\tilde{e}_A \doteq 0 \qquad (4) \Rightarrow$$

$$\Rightarrow y - A\hat{\xi} = [Q_y + (\hat{\xi} \otimes I_n)^T Q_A (\hat{\xi} \otimes I_n)] \cdot \hat{\lambda} =: Q_1 \cdot \hat{\lambda} \qquad (4')$$

$$\Rightarrow \boxed{\hat{\lambda} = Q_1^{-1}(y - A\hat{\xi})}$$ since $Q_1 = Q_1(\hat{\xi})$ is nonsingular.

- Sufficient condition: $\frac{1}{2}\frac{\partial^2 \Phi}{\partial \begin{bmatrix} e_y \\ e_A \end{bmatrix} \partial \left[e_y^T \mid e_A^T \right]} = \begin{bmatrix} P_y & 0 \\ 0 & P_A \end{bmatrix}$ is positive-definite.✓

Fang's Algorithm (2011)

- Nonlinear normal equations from (3) and (4'):

$$\begin{bmatrix} Q_1 & (A - \tilde{E}_A) \\ (A - \tilde{E}_A)^T & 0 \end{bmatrix} \begin{bmatrix} \hat{\lambda} \\ \hat{\xi} \end{bmatrix} = \begin{bmatrix} y - \tilde{E}_A \hat{\xi} \\ 0 \end{bmatrix}$$

with: $\operatorname{vec} \tilde{E}_A = \tilde{e}_A = -Q_A(\hat{\xi} \otimes \hat{\lambda})$ from (2')

- Weighted Total Least-Squares Solution (WTLSS) according to Fang (2011):

$$\hat{\xi} = [(A - \tilde{E}_A)^T Q_1^{-1} (A - \tilde{E}_A)]^{-1} (A - \tilde{E}_A)^T Q_1^{-1} (y - \tilde{E}_A \hat{\xi})$$

where: $\hat{\lambda} = Q_1^{-1}(y - A\hat{\xi})$ from (4')

and: $\operatorname{vec} \tilde{E}_A = \tilde{e}_A = -Q_A(\hat{\xi} \otimes \hat{\lambda})$ from (2')

- Iteration is required!
- Variance component estimate:

$$\tilde{e}_y^T P_y \tilde{e}_y + \tilde{e}_A^T P_A \tilde{e}_A = \hat{\lambda}^T \left[Q_y + (\hat{\xi} \otimes I_n)^T Q_A (\hat{\xi} \otimes I_n) \right] \hat{\lambda} = \hat{\lambda}^T (y - A\hat{\xi}) \Rightarrow$$

$$\Rightarrow \boxed{\hat{\sigma}_0^2 = \hat{\lambda}^T (y - A\hat{\xi})/(n - m)}$$ usually *not* unbiased!

Mahboub's Algorithm (2012)

- Applying the "commutation (or vec-permutation) matrix" K of Magnus and Neudecker (2007) to (3') yields, in conjunction with (2') and (4'):

$$A^T Q_1^{-1}(y - A\hat{\xi}) = A^T\hat{\lambda} = (\hat{\lambda}^T \otimes I_m)\,\text{vec}(\tilde{E}_A^T) = (\hat{\lambda}^T \otimes I_m)K \cdot (\text{vec}\,\tilde{E}_A) =$$

$$= (I_m \otimes \hat{\lambda}^T)\tilde{e}_A = -[(I_m \otimes \hat{\lambda})^T Q_A(\hat{\xi} \otimes Q_1^{-1})] \cdot (y - A\hat{\xi}) =: -R_1 Q_1^{-1} \cdot (y - A\hat{\xi}) \quad \text{(3'')}$$

$$\Rightarrow \boxed{\hat{\xi} = [(A^T + R_1)Q_1^{-1}A]^{-1}(A^T + R_1)Q_1^{-1}y} \quad (7)$$

where : $\hat{\lambda} = Q_1^{-1}(y - A\hat{\xi}), \quad Q_1 := Q_y + (\hat{\xi} \otimes I_n)^T Q_A(\hat{\xi} \otimes I_n) = Q_1(\hat{\xi})$

and : $R_1 := (I_m \otimes \hat{\lambda})^T Q_A(\hat{\xi} \otimes I_n) = R_1(\hat{\xi}, \hat{\lambda})$

- Iteration is required!

- Variance component estimate (not necessarily unbiased):

$$\tilde{e}_y^T P_y \tilde{e}_y + \tilde{e}_A^T P_A \tilde{e}_A = \hat{\lambda}^T Q_1 \hat{\lambda} = \hat{\lambda}^T (y - A\hat{\xi}) \Rightarrow \boxed{\hat{\sigma}_0^2 = \hat{\lambda}^T(y - A\hat{\xi})/(n - m)}$$

- Note: Later, Xu et al. (2012) published a similar algorithm where R_1 has been replaced by $-\tilde{E}_A$, due to $-R_1\hat{\lambda} = \tilde{E}_A\hat{\lambda}$, though $R_1 \neq -\tilde{E}_A$ (in general).

Schaffrin-Wieser Algorithm (2008) when $Q_A = Q_0 \otimes Q_x$

- Introducing a fairly general covariance matrix Q_A with Kronecker product structure:

$$Q_A = \underset{m\times m}{Q_0} \otimes \underset{n\times n}{Q_x} \Rightarrow Q_1 = Q_y + (\hat{\xi}^T Q_0 \hat{\xi}) \cdot Q_x \Rightarrow$$

$$\Rightarrow A^T Q_1^{-1}(y - A\hat{\xi}) = A^T [Q_y + (\hat{\xi}^T Q_0 \hat{\xi}) \cdot Q_x]^{-1}(y - A\hat{\xi}) = A^T \hat{\lambda} = \tilde{E}_A^T \hat{\lambda}$$

with: $\tilde{e}_A = -(Q_0 \hat{\xi} \otimes Q_x)\hat{\lambda} = -\operatorname{vec} \underbrace{(Q_x \hat{\lambda} \hat{\xi}^T Q_0)}_{n\times m} = \operatorname{vec} \tilde{E}_A$

$$\Rightarrow A^T [Q_y + (\hat{\xi}^T Q_0 \hat{\xi}) \cdot Q_x]^{-1}(y - A\hat{\xi}) = -Q_0 \hat{\xi} \cdot (\hat{\lambda}^T Q_x \hat{\lambda}) =: -Q_0 \hat{\xi} \cdot \hat{\nu}$$

with: $\hat{\nu} := (\hat{\lambda}^T Q_x \hat{\lambda})$

and: $\hat{\lambda} := [Q_y + (\hat{\xi}^T Q_0 \hat{\xi}) \cdot Q_x]^{-1}(y - A\hat{\xi})$

- Iteration required!

- Variance component estimate:

$$\tilde{e}_y^T P_y \tilde{e}_y + \tilde{e}_A^T P_A \tilde{e}_A = \hat{\lambda}^T \left[Q_y + (\hat{\xi}^T Q_0 \hat{\xi}) \cdot Q_x\right] \hat{\lambda} = \hat{\lambda}^T (y - A\hat{\xi}) \Rightarrow$$

$\Rightarrow \boxed{\hat{\sigma}_0^2 = \hat{\lambda}^T (y - A\hat{\xi})/(n - m)}$ usually *not* unbiased!

Golub-van-Loan Algorithm (1980) for (diagonal) $Q_x = Q_y$

- For the cofactor matrices, specify $Q_x = Q_y$; then:

$$A^T Q_y^{-1}(y - A\hat{\xi}) = -Q_0\hat{\xi} \cdot \hat{\nu}(1 + \hat{\xi}^T Q_0 \hat{\xi}) =: -Q_0\hat{\xi} \cdot \sigma^2_{\min}$$

with: $\sigma^2_{\min} = (\hat{\lambda}^T Q_y \hat{\lambda})(1 + \hat{\xi}^T Q_0 \hat{\xi}) = (y - A\hat{\xi})^T Q_y^{-1}(y - A\hat{\xi})/(1 + \hat{\xi}^T Q_0 \hat{\xi}) \Rightarrow$

$$\Rightarrow \sigma^2_{\min} \cdot (1 + \hat{\xi}^T Q_0 \hat{\xi}) = y^T Q_y^{-1}(y - A\hat{\xi}) + (\hat{\xi}^T Q_0 \hat{\xi}) \cdot \sigma^2_{\min}$$

$$\Rightarrow \boxed{\sigma^2_{\min} = y^T Q_y^{-1}(y - A\hat{\xi}) = \mathrm{TSSR}}$$

$$\Rightarrow \begin{bmatrix} A^T Q_y^{-1} A & A^T Q_y^{-1} y \\ y^T Q_y^{-1} A & y^T Q_y^{-1} y \end{bmatrix} \begin{bmatrix} \hat{\xi} \\ -1 \end{bmatrix} = \begin{bmatrix} Q_0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\xi} \\ -1 \end{bmatrix} \cdot \sigma^2_{\min}$$

("generalized eigenvalue problem")

- Originally: $Q_0 := I_m$, $Q_y := \mathrm{Diag}(p_1^{-1}, \ldots, p_n^{-1}) = P^{-1}$; then:

$$\begin{bmatrix} A^T P A & A^T P y \\ y^T P A & y^T P y \end{bmatrix} \begin{bmatrix} \hat{\xi} \\ -1 \end{bmatrix} = \begin{bmatrix} \hat{\xi} \\ -1 \end{bmatrix} \cdot \sigma^2_{\min}$$

("standard eigenvalue problem")

- Variance component estimate (not necessarily unbiased):

$$\boxed{\hat{\sigma}_0^2 = \sigma^2_{\min}/(n - m)}$$

The EIV-Model with singular covariance matrices, $\mathrm{rk}[A \mid Q_1] = n$

- Assume that Q_y and/or Q_A are singular, which may have an effect on the *unique solvability* of Fang's normal equation system:

$$\begin{bmatrix} Q_1 & (A - \tilde{E}_A) \\ (A - \tilde{E}_A)^T & 0 \end{bmatrix} \begin{bmatrix} \hat{\lambda} \\ \hat{\xi} \end{bmatrix} = \begin{bmatrix} y - \tilde{E}_A \hat{\xi} \\ 0 \end{bmatrix}, \ \mathrm{vec}\, \tilde{E}_A = \tilde{e}_A = -Q_A(\hat{\xi} \otimes \hat{\lambda});$$

- Since the Least-Squares approach within an iteratively linearized Gauss-Helmert Model generates the *same solution(s)*, the Neitzel-Schaffrin (2013) criterion can be applied to the EIV-Model accordingly in order to ensure the *uniqueness of the TLS solution*. In this case, the criterion to be checked reads:

$$n = \mathrm{rk}\big[BQ \mid A\big] = \mathrm{rk}(BQB^T + ASA^T) =$$
$$= \mathrm{rk}(Q_1 + ASA^T) =: \mathrm{rk}\, Q_2 \quad \text{for } B := \big[I_n \mid -(\xi \otimes I_n)^T\big].$$

  Here, S is a *suitably chosen* symmetric positive-definite matrix (which may have an impact on the convergence speed of an iterative solver).

Generalizing Fang (2011)

- The lower part of Fang's normal equation system yields:

$$(A - \tilde{E}_A)S(A - \tilde{E}_A)^T \cdot \hat{\lambda} = 0$$

- After adding this to the upper part of Fang's system:

$$Q_3 \cdot \hat{\lambda} := \left[Q_1 + (A - \tilde{E}_A)S(A - \tilde{E}_A)^T\right] \cdot \hat{\lambda} = (y - \tilde{E}_A \cdot \hat{\xi}) - (A - \tilde{E}_A) \cdot \hat{\xi} = y - A \cdot \hat{\xi} \Rightarrow$$

$$\Rightarrow 0 = (A - \tilde{E}_A)^T \cdot \hat{\lambda} = (A - \tilde{E}_A)^T Q_3^{-1}\left[(y - \tilde{E}_A \cdot \hat{\xi}) - (A - \tilde{E}_A) \cdot \hat{\xi}\right] \Rightarrow$$

$$\Rightarrow \boxed{\begin{array}{c} \hat{\xi} = [(A - \tilde{E}_A)^T Q_3^{-1}(A - \tilde{E}_A)]^{-1} \cdot (A - \tilde{E}_A)^T Q_3^{-1}(y - \tilde{E}_A \cdot \hat{\xi}) \\ \text{with:} \quad \hat{\lambda} = Q_3^{-1}(y - A\hat{\xi}) \quad \text{and:} \quad \text{vec}\, \tilde{E}_A = \tilde{e}_A = -Q_A \cdot (\hat{\xi} \otimes \hat{\lambda}) \end{array}}$$

provided that $Q_3 := Q_1 + (A - \tilde{E}_A)S(A - \tilde{E}_A)^T$ remains *nonsingular*, $\text{rk}\, Q_3 = \text{rk}\, Q_2 = n$.

- Iteration is required! After convergence:

$$\boxed{\hat{\sigma}_0^2 = \text{TSSR}/(n - m) = (n - m)^{-1} \cdot \hat{\lambda}^T(y - A\hat{\xi})}$$

- Without proof: $\boxed{\hat{D}\{\hat{\xi}\} \approx \hat{\sigma}_0^2 \cdot \left\{\left[(A - \tilde{E}_A)^T Q_3^{-1}(A - \tilde{E}_A)\right]^{-1} - S\right\}}$ as *first order approximation*.

Generalizing Mahboub (2012)

- Without detailed proof, the TLS solution can be obtained from:

$$\boxed{\hat{\xi} = (A^T Q_2^{-1} A + R_2)^{-1} \cdot A^T Q_2^{-1} y}$$

with: $Q_2 := Q_1 + ASA^T = Q_2(\hat{\xi}),\ \text{rk}\, Q_2 = n,$

$R_2 := -(I_m - A^T Q_2^{-1} AS) \cdot \left[(I_m \otimes \hat{\lambda})^T Q_A (I_m \otimes \hat{\lambda})\right] = R_2(\hat{\xi}, \hat{\lambda}),$

and: $\hat{\lambda} = [Q_2 + (AS \otimes \hat{\lambda}^T) Q_A (\hat{\xi} \otimes I_n)]^{-1} \cdot (y - A\hat{\xi}) =$

$= \left[Q_2 + AS \cdot (I_m \otimes \hat{\lambda}^T) Q_A (\hat{\xi} \otimes I_n)\right]^{-1} \cdot (y - A\hat{\xi})$

$$\Rightarrow \boxed{\hat{\lambda} = (Q_2 + AS \cdot R_1)^{-1} \cdot (y - A\hat{\xi})}$$

with: $R_1 := (I_m \otimes \hat{\lambda})^T Q_A (\hat{\xi} \otimes I_n) = R_1(\hat{\xi}, \hat{\lambda})$

- Iteration is required! After convergence:

$$\boxed{\hat{\sigma}_0^2 = \text{TSSR}/(n-m) = (n-m)^{-1} \cdot \hat{\lambda}^T (y - A\hat{\xi})}$$

- Iteratively: $\boxed{\hat{D}\{\hat{\xi}\} \approx \hat{\sigma}_0^2 (A^T Q_2^{-1} A + R_2)^{-1} \cdot A^T Q_2^{-1} \cdot Q_y \cdot Q_2^{-1} A \cdot (A^T Q_2^{-1} A + R_2^T)^{-1}}$
  as *first order approximation*.

The new family of weighted TLS algorithms I

(i) Assuming that $C\{e_y, e_A\} = 0$ (as so far)

- From the generalized Mahboub algorithm:

$$A^T Q_2^{-1}(y - A\hat{\xi}) = R_2 \cdot \hat{\xi} = (A^T Q_2^{-1} AS - I_m) \cdot \left[(I_m \otimes \hat{\lambda})^T Q_A (I_m \otimes \hat{\lambda})\right] \cdot \hat{\xi}$$

$$\Rightarrow \boxed{\begin{array}{c} A^T Q_2^{-1}\left[y - AS \cdot (I_m \otimes \hat{\lambda})^T Q_A (I_m \otimes \hat{\lambda}) \cdot \hat{\xi}\right] = \\ = \left[A^T Q_2^{-1} A - (I_m \otimes \hat{\lambda})^T Q_A (I_m \otimes \hat{\lambda})\right] \cdot \hat{\xi} \end{array}}$$

to be solved *iteratively* in conjunction with

$$\boxed{(Q_2 + AS \cdot R_1) \cdot \hat{\lambda} = \left[Q_2 + AS \cdot (I_m \otimes \hat{\lambda})^T Q_A (\hat{\xi} \otimes I_n)\right] \cdot \hat{\lambda} = y - A\hat{\xi}}$$

- Variance component estimate:

$$\boxed{\hat{\sigma}_0^2 = \text{TSSR}/(n - m) = (n - m)^{-1} \cdot \hat{\lambda}^T (y - A\hat{\xi})}$$

- No new formula for the estimated dispersion matrix $\hat{D}\{\hat{\xi}\}$ yet.

The new family of weighted TLS algorithms II

(ii) In case of non-negligible cross-covariances, $C\{e_y, e_A\} = \sigma_0^2 Q_{yA} \neq 0$

- From Snow (2012, §3.2), for $Z := (I_m \otimes \hat{\lambda})^T Q_{yA}$:

$$(A - Z)^T (Q_2')^{-1}(y - A\hat{\xi}) = R_2' \cdot \hat{\xi} =$$
$$= \left[(A - Z)^T (Q_2')^{-1}(A - Z)S - I_m\right] \cdot \left[(I_m \otimes \hat{\lambda})^T Q_A (I_m \otimes \hat{\lambda})\right] \cdot \hat{\xi}$$

where $\quad Q_2' := Q_1' + (A - Z)S(A - Z)^T$

and $\quad Q_1' := Q_1 - Q_{yA}(\hat{\xi} \otimes I_n) - (\hat{\xi} \otimes I_n)^T Q_{Ay}$

$$\Rightarrow \boxed{\begin{aligned} &(A - Z)^T \left(Q_2'\right)^{-1} \left[(y - Z\hat{\xi}) - (A - Z)S \cdot (I_m \otimes \hat{\lambda})^T Q_A (I_m \otimes \hat{\lambda}) \cdot \hat{\xi}\right] = \\ &= \left[(A - Z)^T \left(Q_2'\right)^{-1}(A - Z) - (I_m \otimes \hat{\lambda})^T Q_A (I_m \otimes \hat{\lambda})\right] \cdot \hat{\xi} \end{aligned}}$$

to be solved *iteratively* in conjunction with

$$\boxed{\left[Q_2' + (A - Z)S{\cdot}R_1\right]{\cdot}\hat{\lambda} = \left[Q_2' + (A - Z)S{\cdot}(I_m \otimes \hat{\lambda})^T Q_A (\hat{\xi} \otimes I_n)\right]{\cdot}\hat{\lambda} = (y - Z\hat{\xi}) - (A - Z)\hat{\xi}}$$

- Variance component estimate:

$$\boxed{\hat{\sigma}_0^2 = \mathrm{TSSR}/(n - m) = (n - m)^{-1}{\cdot}\hat{\lambda}^T\left[(y - Z\hat{\xi}) - (A - Z)\hat{\xi}\right] = (n - m)^{-1}{\cdot}\hat{\lambda}^T(y - A\hat{\xi})}$$

New algorithmic scheme

New algorithm (suggested):

Step 1: Compute an initial solution

$$\hat{\xi}^{(0)} := N^{+}c \quad \text{for } [N \mid c] := A^T Q_y^{+} [A \mid y], \quad \hat{\lambda}^{(0)} = Q_y^{+}\left(y - A\hat{\xi}^{(0)}\right),$$

Step 2: For $i \in \mathbb{N}$ and a chosen matrix S, compute

$$Z^{(i)} = \left(I_m \otimes \hat{\lambda}^{(i-1)}\right)^T Q_{yA},$$

$$Q'^{(i)}_1 = Q_y + \left(\hat{\xi}^{(i-1)} \otimes I_n\right)^T Q_A \left(\hat{\xi}^{(i-1)} \otimes I_n\right) - Q_{yA}\left(\hat{\xi}^{(i-1)} \otimes I_n\right) - \left(\hat{\xi}^{(i-1)} \otimes I_n\right)^T Q_{Ay},$$

$$Q'^{(i)}_2 = Q'^{(i)}_1 + (A - Z^{(i)})S(A - Z^{(i)})^T,$$

$$R_1^{(i)} = \left(I_m \otimes \hat{\lambda}^{(i-1)}\right)^T Q_A \left(\hat{\xi}^{(i-1)} \otimes I_n\right),$$

$$\hat{\lambda}^{(i)} = \left[Q'^{(i)}_2 + (A - Z^{(i)})S \cdot R_1^{(i)}\right]^{-1} \cdot \left(y - A\hat{\xi}^{(i-1)}\right),$$

$$\hat{\xi}^{(i)} = \left[(A - Z^{(i)})^T \left(Q'^{(i)}_2\right)^{-1} (A - Z^{(i)}) - \left(I_m \otimes \hat{\lambda}^{(i)}\right)^T Q_A \left(I_m \otimes \hat{\lambda}^{(i)}\right)\right]^{-1} \cdot$$

$$\cdot (A - Z^{(i)})^T \left(Q'^{(i)}_2\right)^{-1} \cdot \left[\left(y - Z^{(i)}\hat{\xi}^{(i-1)}\right) - (A - Z^{(i)})^T S \cdot (I_m \otimes \hat{\lambda}^{(i)})^T Q_A (I_m \otimes \hat{\lambda}^{(i)}) \cdot \hat{\xi}^{(i-1)}\right]$$

Step 3: Stop when $\left\|\hat{\lambda}^{(i)} - \hat{\lambda}^{(i-1)}\right\| < \delta$ and $\left\|\hat{\xi}^{(i)} - \hat{\xi}^{(i-1)}\right\| < \delta$ for a chosen threshold δ; then compute

$$\hat{\sigma}_0^2 = (n - m)^{-1} \cdot \hat{\lambda}^T (y - A\hat{\xi}).$$

Conclusions and outlook

- The original Mahboub (2012) algorithm has been modified in such a way that, for the computation of $\hat{\xi}$, a *symmetric* system needs to be solved that has turned out *positive-definite* so far.
- It is unclear, however, whether this matrix is *necessarily* nonnegative-definite (if not positive-definite).
- In the limited number of examples considered so far, the new algorithm proved *faster and more efficient* than the original Mahboub algorithm.
- A *more systematic* comparison, that also includes the generalized version of Fang's (2011) algorithm, still needs to be completed.

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