Links between $\boxed{\text{fixed}}$ linear model and $\boxed{\text{mixed}}$ linear model

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- The main result is the fact that a **mixed** linear model can be obtained as a transformation of the extended linear regression model with **fixed** effects.
- This observation is simple, but we are not aware of it being previously mentioned in the literature.
- Why should it be of interest?

- **Why should it be of interest?**

- Simo:

  Get

  going,

  young man!

Abstract

- The linear **mixed** model has strong links with a particular partitioned linear model including only **fixed** effects.

Abstract

- The linear mixed model has strong links with a particular partitioned linear model including only fixed effects.

- The connection between the two models is very straightforward:

  —the mixed model can be obtained from the fixed effects model by a simple linear transformation.

We play with three different models:

- $\mathscr{A} = \{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \mathbf{V}\}$: $\boxed{\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}}$

- $\mathscr{M} = \{\mathbf{y}, \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{u}, \mathbf{D}, \mathbf{R}\}$: $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{u} + \boldsymbol{\varepsilon}$

- $\mathscr{F}_* = \{\mathbf{y}_*, \mathbf{X}_*\boldsymbol{\pi}, \mathbf{V}_*\}$

$$= \left\{ \begin{pmatrix} \mathbf{y} \\ \mathbf{y}_0 \end{pmatrix}, \begin{pmatrix} \mathbf{X} & \mathbf{Z} \\ \mathbf{0} & -\mathbf{I}_q \end{pmatrix} \begin{pmatrix} \boldsymbol{\beta} \\ \boldsymbol{\gamma} \end{pmatrix}, \begin{pmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} \end{pmatrix} \right\}$$

$\mathscr{M}$ *vs* $\mathscr{F}_*$ **BLUP**$(\mathbf{u})$ *vs* **BLUE**$(\boldsymbol{\gamma})$

1. Mixed model $\mathscr{M}$

$$\mathscr{M}: \quad \mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{u} + \boldsymbol{\varepsilon}\,,$$

- $\mathbf{y}$ is an $n \times 1$ observable random vector,
- $\boldsymbol{\varepsilon}$ is an $n \times 1$ random error vector,
- $\mathbf{X}$, $\mathbf{Z}$ are known $n \times p$ and $n \times q$ matrices,
- $\boldsymbol{\beta}$ **is a** $p \times 1$ **vector of unknown parameters,**
- $\mathbf{u}$ **is a** $q \times 1$ **unobservable vector of** *random effects.*

$$\mathrm{E}(\mathbf{u}) = \mathbf{0}, \quad \operatorname{cov}(\mathbf{u}) = \mathbf{D},$$

$$\mathrm{E}(\boldsymbol{\varepsilon}) = \mathbf{0}, \quad \operatorname{cov}(\boldsymbol{\varepsilon}) = \mathbf{R},$$

$$\operatorname{cov}(\boldsymbol{\varepsilon}, \mathbf{u}) = \mathbf{0},$$

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and $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{u} + \boldsymbol{\varepsilon}$ implies that

$$\operatorname{cov}(\mathbf{y}) = \mathbf{Z}\mathbf{D}\mathbf{Z}' + \mathbf{R} := \boldsymbol{\Sigma}.$$

Denote this mixed model as

$$\mathscr{M} = \{\mathbf{y},\, \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{u},\, \mathbf{D},\, \mathbf{R}\}\,.$$

If $\mathbf{Ay}$ is a **LUP** for $\mathbf{u}$, then $\mathbf{Ay}$ is the **BLUP** for $\mathbf{u}$ if

$$\operatorname{cov}(\mathbf{Ay} - \mathbf{u}) \leq_{\mathrm{L}} \operatorname{cov}(\mathbf{Ny} - \mathbf{u})$$

for all $\mathbf{N} : \ \mathbf{Ny} = \mathbf{LUP}(\mathbf{u})$.

2. Three lemmas

Conditions under which

$$\mathbf{A}\mathbf{y} = \mathbf{BLUP}(\mathbf{u}),$$

$$\mathbf{G}\mathbf{y} = \mathbf{BLUE}(\mathbf{X}\boldsymbol{\beta}).$$

Lemma 1 *Under the mixed model* $\mathscr{M} = \{\mathbf{y},\, \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{u},\, \mathbf{D},\, \mathbf{R}\}$:

(a) $\mathbf{A}\mathbf{y} = \mathbf{BLUP}(\mathbf{u})$,

(b) $\mathbf{A}(\mathbf{X} : \boldsymbol{\Sigma}\mathbf{X}^{\perp}) = (\mathbf{0} : \mathbf{D}\mathbf{Z}'\mathbf{X}^{\perp})$.

LEMMA **2** *Under* $\mathscr{A} = \{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \mathbf{V}\}$*:*

$$\text{(a) } \mathbf{G}\mathbf{y} = \mathbf{BLUE}(\mathbf{X}\boldsymbol{\beta}),$$

$$\text{(b) } \mathbf{G}(\mathbf{X} : \mathbf{V}\mathbf{X}^{\perp}) = (\mathbf{X} : \mathbf{0}).$$

$$\text{(c) } \mathbf{B}\mathbf{y} = \mathbf{BLUP}(\mathbf{L}\boldsymbol{\varepsilon}),$$

$$\text{(d) } \mathbf{B}(\mathbf{X} : \mathbf{V}\mathbf{X}^{\perp}) = (\mathbf{0} : \mathbf{L}\mathbf{V}\mathbf{X}^{\perp}).$$

That was all folklore but

the 3rd lemma

is

of different style . . .

Lemma 3 *The equality*

$$\mathbf{BLUP}(\mathbf{L}\boldsymbol{\varepsilon} \mid \mathscr{A}) = \mathbf{Ly} - \mathbf{BLUE}(\mathbf{LX}\boldsymbol{\beta} \mid \mathscr{A})$$

holds w.p. 1, that is, for every $\mathbf{y} \in \mathscr{C}(\mathbf{X} : \mathbf{VX}^{\perp})$.

3. Partitioned model: some properties

. . .

Consider

$$\mathscr{A}_{12} = \{\mathbf{y}, \mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\boldsymbol{\beta}_2, \mathbf{V}\}$$

and its **transformed version**

$$\mathscr{A}_{\mathbf{F}} = \{\mathbf{F}'\mathbf{y}, \mathbf{F}'\mathbf{X}_1\boldsymbol{\beta}_1, \mathbf{F}'\mathbf{V}\mathbf{F}\},$$

where $\mathbf{F}'\mathbf{X}_2 = \mathbf{0}$. The Frisch–Waugh–Lovell thm deals with $\mathbf{F} = \mathbf{M}_2 = \mathbf{I}_n - \mathbf{P}_{\mathbf{X}_2}$. Denote

$$\mathscr{A}_{12\cdot 2} = \{\mathbf{M}_2\mathbf{y}, \mathbf{M}_2\mathbf{X}_1\boldsymbol{\beta}_1, \mathbf{M}_2\mathbf{V}\mathbf{M}_2\}.$$

According to FWL:

$$\{\mathbf{BLUE}(\mathbf{M}_2\mathbf{X}_1\boldsymbol{\beta}_1 \mid \mathscr{A}_{12})\} = \{\mathbf{BLUE}(\mathbf{M}_2\mathbf{X}_1\boldsymbol{\beta}_1 \mid \mathscr{A}_{12\cdot 2})\}.$$

The next slide provides two new results . . .

LEMMA 4 *Let* $\mathbf{F} \in \{\mathbf{X}_2^{\perp}\}$. *Then*

$$\{\mathbf{BLUE}(\mathbf{F}'\mathbf{X}_1\boldsymbol{\beta}_1 \mid \mathscr{A}_{12})\}$$
$$= \{\mathbf{BLUE}(\mathbf{F}'\mathbf{X}_1\boldsymbol{\beta}_1 \mid \mathscr{A}_{\mathbf{F}})\}.$$

$$\{\mathbf{BLUP}(\mathbf{L}\boldsymbol{\varepsilon} \mid \mathscr{A}_{12})\}$$
$$= \{\mathbf{BLUP}(\mathbf{L}\boldsymbol{\varepsilon} \mid \mathscr{A}_{\mathbf{F}})\}.$$

IWMS-Montréal, 1995

4. Transforming the big model $\mathscr{F}_*$

Consider an extended "big" model

$$\mathscr{F}_* = \{\mathbf{y}_*, \mathbf{X}_*\boldsymbol{\pi}, \mathbf{V}_*\}$$

$$= \left\{ \begin{pmatrix} \mathbf{y} \\ \mathbf{y}_0 \end{pmatrix}, \begin{pmatrix} \mathbf{X} & \mathbf{Z} \\ \mathbf{0} & -\mathbf{I}_q \end{pmatrix} \begin{pmatrix} \boldsymbol{\beta} \\ \boldsymbol{\gamma} \end{pmatrix}, \begin{pmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} \end{pmatrix} \right.$$

where both $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ are non-random:

Consider a partitioned extended "big" model

$$
\begin{aligned}
\mathscr{F}_* &= \{\mathbf{y}_*, \mathbf{X}_*\boldsymbol{\pi}, \mathbf{V}_*\} \\
&= \left\{ \begin{pmatrix} \mathbf{y} \\ \mathbf{y}_0 \end{pmatrix}, \begin{pmatrix} \mathbf{X} & \mathbf{Z} \\ \mathbf{0} & -\mathbf{I}_q \end{pmatrix} \begin{pmatrix} \boldsymbol{\beta} \\ \boldsymbol{\gamma} \end{pmatrix}, \begin{pmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} \end{pmatrix} \right\},
\end{aligned}
$$

where both $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ are regarded as non-random:

$$
\begin{aligned}
\mathbf{y} &= \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\boldsymbol{\gamma} + \boldsymbol{\varepsilon}, \\
\mathbf{y}_0 &= \qquad\quad -\boldsymbol{\gamma} + \boldsymbol{\varepsilon}_0.
\end{aligned}
$$

$$\begin{cases} \mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\boldsymbol{\gamma} + \boldsymbol{\varepsilon}, & \text{(a)} \\ \mathbf{y}_0 = \qquad\qquad -\boldsymbol{\gamma} + \boldsymbol{\varepsilon}_0. & \text{(b)} \end{cases}$$

Premultiplying (a) by $\mathbf{Z}$ and adding the result into (b) yields

$$\begin{aligned} \mathbf{y} + \mathbf{Z}\mathbf{y}_0 &= \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon} + \mathbf{Z}\boldsymbol{\varepsilon}_0 \\ &= \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\boldsymbol{\varepsilon}_0 + \boldsymbol{\varepsilon}. \qquad \text{(c)} \end{aligned}$$

Now we see an interesting development:

$$\boxed{\mathbf{y} + \mathbf{Z}\mathbf{y}_0 = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\boldsymbol{\varepsilon}_0 + \boldsymbol{\varepsilon}}$$

defines **a mixed model**

$$\mathscr{T} = \{\mathbf{w}, \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\boldsymbol{\varepsilon}_0, \mathbf{D}, \mathbf{R}\},$$

where the response is $\mathbf{w} = \mathbf{y} + \mathbf{Z}\mathbf{y}_0$ and $\boldsymbol{\varepsilon}_0$ is the random effect.

Wasn't it interesting?

In matrix terms, premultiplying

$$\mathscr{F}_* = \{\mathbf{y}_*,\ \mathbf{X}_*\boldsymbol{\pi},\ \mathbf{V}_*\}$$

by $\mathbf{T}' = (\mathbf{I}_n : \mathbf{Z})$ gives

$$\mathbf{T}'\mathbf{y}_* = \mathbf{y} + \mathbf{Z}\mathbf{y}_0\,,$$

$$\mathbf{T}'\mathbf{X}_* = (\mathbf{X} : \mathbf{0})\,,$$

$$\mathbf{T}'\mathbf{V}_*\mathbf{T} = \mathbf{R} + \mathbf{Z}\mathbf{D}\mathbf{Z}' = \boldsymbol{\Sigma},$$

and so the new resulting model is

$$\begin{aligned}
\mathscr{T} &= \{\mathbf{T}'\mathbf{y}_*,\ \mathbf{T}'\mathbf{X}_*\boldsymbol{\pi},\ \mathbf{T}'\mathbf{V}_*\mathbf{T}\} \\
&= \{\mathbf{y} + \mathbf{Z}\mathbf{y}_0,\ \mathbf{X}\boldsymbol{\beta},\ \mathbf{Z}\mathbf{D}\mathbf{Z}' + \mathbf{R}\} \\
&= \{\mathbf{w},\ \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\boldsymbol{\varepsilon}_0,\ \boldsymbol{\Sigma}\} \\
&= \text{ a mixed model.}
\end{aligned}$$

5. The connection between $\mathscr{F}_*$ and $\mathscr{T}$

THEOREM **1** *With probability* 1:

$$\mathbf{BLUE}(\mathbf{X}\boldsymbol{\beta} \mid \mathscr{F}_*) = \mathbf{BLUE}(\mathbf{X}\boldsymbol{\beta} \mid \mathscr{T}).$$

THEOREM **2** *With probability* 1:

$$\mathbf{BLUP}(\boldsymbol{\varepsilon}_0 \mid \mathscr{T}) = \mathbf{y}_0 + \mathbf{BLUE}(\boldsymbol{\gamma} \mid \mathscr{F}_*).$$

$$\mathscr{F}_* = \{\mathbf{y}_*, \mathbf{X}_*\boldsymbol{\pi}, \mathbf{V}_*\}$$
$$= \left\{ \begin{pmatrix} \mathbf{y} \\ \mathbf{y}_0 \end{pmatrix}, \begin{pmatrix} \mathbf{X} & \mathbf{Z} \\ \mathbf{0} & -\mathbf{I}_q \end{pmatrix} \begin{pmatrix} \boldsymbol{\beta} \\ \boldsymbol{\gamma} \end{pmatrix}, \begin{pmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} \end{pmatrix}\right.$$

With probability 1:

$$\mathbf{BLUP}(\boldsymbol{\varepsilon}_0 \mid \mathscr{T}) = \mathbf{y}_0 + \mathbf{BLUE}(\boldsymbol{\gamma} \mid \mathscr{F}_*).$$

$$\mathscr{F}_*: \qquad \mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\boldsymbol{\gamma} + \boldsymbol{\varepsilon},$$

$$\mathbf{y}_0 = \qquad\qquad -\boldsymbol{\gamma} + \boldsymbol{\varepsilon}_0.$$

$$\mathscr{T} = \{\mathbf{w}, \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\boldsymbol{\varepsilon}_0, \mathbf{D}, \mathbf{R}\}.$$

With probability 1:

$$\mathbf{BLUP}(\boldsymbol{\varepsilon}_0 \mid \mathscr{T}) = \mathbf{y}_0 + \mathbf{BLUE}(\boldsymbol{\gamma} \mid \mathscr{F}_*)$$

Put $\mathbf{y}_0 = \mathbf{0}$, then

$$\boxed{\mathbf{BLUP}(\boldsymbol{\varepsilon}_0 \mid \mathscr{T}) = \mathbf{BLUE}(\boldsymbol{\gamma} \mid \mathscr{F}_*).}$$

- Having an observed response $\mathbf{w}$ in the mixed linear model $\mathscr{T}$, ...

- construct $\mathbf{y}$ for $\mathscr{F}_*$ by taking any $\mathbf{y}_0$ and putting $\mathbf{y} = \mathbf{w} - \mathbf{Z}\mathbf{y}_0$,

- obtain the **BLUP** for random effects under $\mathscr{T}$ as $\mathbf{y}_0 + \mathbf{BLUE}(\boldsymbol{\gamma} \mid \mathscr{F}_*)$ using the appropriate $\mathbf{y}$ and $\mathbf{y}_0$.

- Choosing $\mathbf{y}_0 = \mathbf{0}$, yields the famous <u>Henderson's</u> result

$$\boxed{\mathbf{BLUP}(\boldsymbol{\varepsilon}_0 \mid \mathscr{T}) = \mathbf{BLUE}(\boldsymbol{\gamma} \mid \mathscr{F}_*)}$$

where the response vector in $\mathscr{F}_*$ is $\begin{pmatrix} \mathbf{w} \\ \mathbf{0} \end{pmatrix}$.

0. Introduction

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Smolenice Castle, 2011