Studying the singularity of LCM-type matrices via semilattice structures and their Möbius functions

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Basic assumptions and definitions

- Let $S = \{x_1, x_2, \ldots, x_n\}$ be a finite and GCD closed subset of $\mathbb{Z}_+$ with $x_1 < x_2 < \cdots < x_n$
  - i.e., $\gcd(x_i, x_j) \in S$ for all $i, j \in \{1, 2, \ldots, n\}$
- Let α be a positive real number
- The $n \times n$ matrix having $(\gcd(x_i, x_j))^\alpha$ as its ij entry is the GCD matrix of the set S raised to the Hadamard power of α (shortly power GCD matrix) and is denoted by $(S)_{N^\alpha}$
- The $n \times n$ matrix having $(\mathrm{lcm}(x_i, x_j))^\alpha$ as its ij entry is the LCM matrix of the set S raised to the Hadamard power of α (shortly power LCM matrix) and is denoted by $[S]_{N^\alpha}$
- In the special case when $\alpha = 1$ we obtain the usual GCD and LCM matrices of the set S, which are denoted by (S) and $[S]$, respectively

The Bourque-Ligh Conjecture

In 1992 Bourque and Ligh presented the following conjecture:

Conjecture

The LCM matrix of a GCD closed set S is always invertible.

Later it has been shown that

- The B-L-Conjecture holds for GCD closed sets S with less than 8 elements (i.e. the conjecture holds for $n \leq 7$)
- For every $n \geq 8$ there exists a GCD closed set S such that $|S| = n$ and the LCM matrix $[S]$ is singular (i.e. the conjecture does not hold generally for $n \geq 8$)

This raises a couple of new questions:

- It is not sufficient to assume that the set S is GCD closed in order to guarantee the invertibility of the matrix $[S]$, what further assumptions are needed?
- What conditions guarantee the invertibility of the power LCM matrix $[S]_{N^\alpha}$ for all $\alpha > 0$?

Sufficient and necessary condition for singularity

Since

$$[S]_{N^\alpha} = \mathrm{diag}(x_1^\alpha, \ldots, x_n^\alpha)(S)_{\frac{1}{N^\alpha}}\mathrm{diag}(x_1^\alpha, \ldots, x_n^\alpha),$$

it follows that $[S]_{N^\alpha}$ is singular if and only if $(S)_{\frac{1}{N^\alpha}}$ is singular. Further, since the set S is GCD closed, we may define the function $\Psi_{S,\frac{1}{N^\alpha}}$ on S as

$$\Psi_{S,\frac{1}{N^\alpha}}(x_i) = \sum_{x_k \mid x_i} \frac{\mu_S(x_k, x_i)}{x_k^\alpha} \tag{1.1}$$

By a well-known determinant formula we now have

$$\det(S)_{\frac{1}{N^\alpha}} = \Psi_{S,\frac{1}{N^\alpha}}(x_1)\Psi_{S,\frac{1}{N^\alpha}}(x_2)\cdots\Psi_{S,\frac{1}{N^\alpha}}(x_n). \tag{1.2}$$

Thus we may conclude the following result.

Proposition

The matrices $[S]_{N^\alpha}$ and $(S)_{\frac{1}{N^\alpha}}$ are both invertible if and only if $\Psi_{S,\frac{1}{N^\alpha}}(x_i) \neq 0$ for all $i = 1, \ldots, n$.

An auxiliary lemma

The following lemma is going to tell us something important about the zeros of the Möbius function of a finite meet semilattice.

Lemma

Let $(L, \preceq)$ *be a finite meet semilattice,* $x \in L$ *and* $C_L(x) = \{y \in L \mid x \text{ covers } y\}$. *Denote* $\xi_L(x) = \bigwedge C_L(x)$ *if* $C_L(x) \neq \varnothing$ *and* $\xi_L(x) = x$ *if* $C_L(x) = \varnothing$. *If*

$$z \notin [\![\xi_L(x), x]\!] := \{w \in L \mid \xi_L(x) \preceq w \preceq x\},$$

then $\mu_L(z, x) = 0$.

Proof.

The proof is left as a cumbersome exercise (requires double induction: first on the size of $C_L(x)$ and then on the size of the interval $[\![z, \xi_{[\![z,x]\!]}(x)[\![$.) □

Conjecture on singularity of LCM matrices

It can be shown that if S is an odd GCD closed set with at most 8 elements, then the LCM matrix $[S]$ is always nonsingular (an odd set is a set whose all elements are odd). Hong took this idea even further by presenting the following conjecture.

Conjecture (Hong, J. Number Theory, 2005)

The LCM matrix $[S]$ defined on any odd GCD closed set S is nonsingular.

However, this conjecture fails already when $|S| = 9$.

Theorem

The above conjecture does not hold.

Proof.

Let us consider the odd set

$$S = \{1, 3, 5, 7, 195, 291, 1407, 4025, 1020180525\} =$$

$$\{1, 3, 5, 7, 3 \cdot 5 \cdot 13, 3 \cdot 97, 3 \cdot 7 \cdot 67, 5^2 \cdot 7 \cdot 23, 3 \cdot 5^2 \cdot 7 \cdot 13 \cdot 23 \cdot 67 \cdot 97\}.$$

The proof continues

We obtain

$$\Psi_{S,\frac{1}{N}}(1020180525)$$
$$= \frac{1}{1020180525} - \frac{1}{4025} - \frac{1}{1407} - \frac{1}{291} - \frac{1}{195} + \frac{1}{7} + \frac{1}{5} + \frac{2}{3} - 1 = 0,$$

and thus the matrix $[S]$ is singular. □

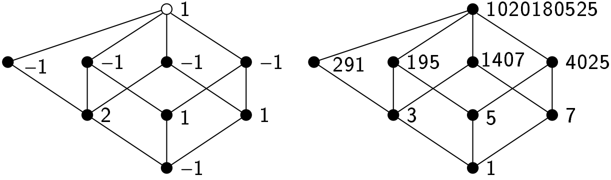


Figure: The Hasse diagram of the counterexample. The left figure shows the values $\mu_S(x_i, x_9)$, the right shows the respective elements of S.

Odd singular numbers

A positive integer x is said to be *a singular number* if there exists a GCD closed set $S = \{x_1, \ldots, x_n\}$, where $1 \leq x_1 < \cdots < x_n = x$, such that $\Psi_{S,\frac{1}{N}}(x) = 0$. Otherwise x is a *nonsingular number*. Moreover, x is a *primitive singular number* if x is singular and x' is nonsingular number for all $x' \,|\, x$, $x' \neq x$. Hong conjectured that there are infinitely many even primitive singular numbers. About odd primitive singular numbers he cojectured the following:

Conjecture (Hong, J. Number Theory, 2005)

There does not exist an odd primitive singular number.

The earlier counterexample disproves also this conjecture.

Corollary

There exists an odd primitive singular number.

Proof.

We know that 1020180525 is an odd singular number. If it is not primitive singular number itself, then it has a nontrivial factor which is an odd primitive singular number. □

The case with real exponent

So far we have been studying the singularity of the usual LCM matrices (with $\alpha = 1$).

How much easier would it be to find semilattice structures which yield singular power LCM matrices, when the exponent α is allowed to be any *positive real* number?

We begin our study with two illustrative examples.

Example: chain semilattice

Example

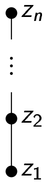
Let $L = \{z_1, z_2, \ldots, z_n\}$ be a chain with $z_1 < z_2 < \cdots < z_n$, let α be any positive real number and let S be any set of positive integers such that $(S, |) \cong (L, \leq)$. Then we get

$$\Psi_{S,\frac{1}{N^\alpha}}(x_1) = \frac{\mu_S(x_1, x_1)}{x_1^\alpha} = \frac{1}{x_1^\alpha} > 0,$$

and for $1 < i \leq n$ we have

$$\Psi_{S,\frac{1}{N^\alpha}}(x_i) = \frac{1}{x_i^\alpha} - \frac{1}{x_{i-1}^\alpha} < 0.$$

Thus $[S]_{N^\alpha} = [\mathrm{lcm}(x_i, x_j)^\alpha]$ is invertible for all $\alpha > 0$.



Example: diamond semilattice

Example

Let $(L, \preceq)$ be the four element diamond meet semilattice. Suppose that $S = \{x_1, x_2, x_3, x_4\} = \{1, 3, 5, 45\}$. Let α be any positive real number. Applying (1.1) we obtain

$$\Psi_{S,\frac{1}{N^\alpha}}(1) = 1, \quad \Psi_{S,\frac{1}{N^\alpha}}(3) = \frac{1}{3^\alpha} - 1 \quad \text{and} \quad \Psi_{S,\frac{1}{N^\alpha}}(5) = \frac{1}{5^\alpha} - 1,$$

which are all nonzero for all $\alpha > 0$. However,

$$\Psi_{S,\frac{1}{N^\alpha}}(45) = \frac{1}{45^\alpha} - \frac{1}{5^\alpha} - \frac{1}{3^\alpha} + 1,$$

which is negative for $\alpha = \frac{1}{4}$ and positive for $\alpha = 1$. Since $\Psi_{S,\frac{1}{N^\alpha}}(45)$ is a continuous function of variable α, this function must have zero value for some positive α_0 (this α_0 is located approximately at 0.328594). It follows that the matrix $[S]_{N^{\alpha_0}} = [[x_i, x_j]^{\alpha_0}]$ is singular. Thus the diamond structure does not possess the same property as chains were proven to have in our previous example.

Example: diamond semilattice

Example

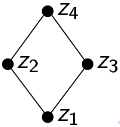
Although we just found one set S that yields a singular power LCM matrix for some positive real number α, not every set of positive integers isomorphic to $(L, \preceq)$ has this property. To see this we only need to choose $S' = \{x_1', x_2', x_3', x_4'\} = \{1, 3, 5, 15\}$. In this case we have

$$\Psi_{S', \frac{1}{N^\alpha}}(i) = \Psi_{S, \frac{1}{N^\alpha}}(i) \neq 0 \quad \text{for all } \alpha > 0 \text{ and for all } i = 1, 2, 3,$$

but also

$$\Psi_{S', \frac{1}{N^\alpha}}(15) = \frac{1}{15^\alpha} - \frac{1}{5^\alpha} - \frac{1}{3^\alpha} + 1 = \frac{1}{15^\alpha}(5^\alpha - 1)(3^\alpha - 1) \neq 0$$

for all $\alpha > 0$. This means that the power LCM matrix $[S']_{N^\alpha} = [[x_i', x_j']^\alpha]$ is nonsingular for all $\alpha > 0$.



A theorem on how to find singular power LCM matrices

Sometimes the lattice-theoretic structure of $(S, |)$ alone tells us that the power LCM matrix of the set S is invertible for all $\alpha > 0$. On the other hand, in the remaining cases the information about the structure of $(S, |)$ is inconclusive and does not reveal whether or not all the power LCM matrices of the set S are invertible. Our ultimate goal is to characterize all possible meet semilattices $(L, \preceq)$, whose structure is strong enough to guarantee the invertibility of the power LCM matrix for all GCD closed set $(S, |) \cong (L, \preceq)$ and for all $\alpha > 0$.

Theorem

Let $(L, \preceq)$ be a meet semilattice with n elements. Assume that there exist elements $x, y_1, \ldots, y_m$ $(m \geq 2)$ in L such that x covers $y_1, \ldots, y_m$ and $\mu_L(y, x) > 0$, where $y = y_1 \wedge \cdots \wedge y_m$. Then there exists a set $S = \{x_1, x_2, \ldots, x_n\}$ of positive integers and a positive real number α_0 such that $(S, |) \cong (L, \preceq)$ and the power LCM matrix $[S]_{N^{\alpha_0}} = [[x_i, x_j]^{\alpha_0}]$ of the set S is singular.

The proof

Let us denote $L = \{z_1, \ldots, z_n\}$, where $z_i \leq z_j \Rightarrow i \leq j$ (in particular, $z_1 = \min L$). We begin by constructing a GCD closed set $S' = \{x'_1, x'_2, \ldots, x'_n\}$ of positive integers such that $(S', |) \cong (L, \leq)$. Let $p_2, p_3, \ldots, p_n$ be distinct prime numbers. We define $x'_1 = 1$ and

$$x'_i = p_i \mathrm{lcm}\{x'_j \,\big|\, j < i \text{ and } z_j \leq z_i\} = \prod_{\substack{1 \leq j \leq i \\ z_j \leq z_i}} p_j$$

for $1 < i \leq n$. It is easy to see that the set S' is both GCD closed and isomorphic to L (every element of S' is either 1 or a squarefree product of different primes).

Now suppose that $x'_i \in S'$ is an element such that it covers the elements $x'_{i_1}, x'_{i_2}, \ldots, x'_{i_m} \in S'$ and $\mu_{S'}(x'_k, x'_i) > 0$, where $x'_k = x'_{i_1} \wedge x'_{i_2} \wedge \cdots \wedge x'_{i_m}$. Let r be an arbitrary positive integer. Now let $S(r) = \{x_1, x_2, \ldots, x_n\}$, where

$$x_j = \begin{cases} x'_j & \text{if } x'_i \nmid x'_j, \\ p_i^r x'_j & \text{if } x'_i \,|\, x'_j. \end{cases}$$

Clearly $(S(r), |) \cong (S', |) \cong (L, \leq)$.

The proof continues

Let i be as fixed above. Then $x_i = p_i^r x_i'$. Let r be sufficiently large (to be specified later). We define the function $h_{i,r} : \mathbb{R} \to \mathbb{R}$ by

$$h_{i,r}(\alpha) = \Psi_{S(r),\frac{1}{N^\alpha}}(x_i) = \sum_{j=1}^{i} \frac{\mu_{S(r)}(x_j, x_i)}{x_j^\alpha}.$$

By Lemma we know that $\mu_{S(r)}(x_j, x_i) = 0$ for all $x_j \notin [\![x_k, x_i]\!]$. Thus the function $h_{i,r}$ comes to the form

$$h_{i,r}(\alpha) = \sum_{x_k \mid x_j \mid x_i} \frac{\mu_{S(r)}(x_j, x_i)}{x_j^\alpha} = \frac{1}{x_k^\alpha} \sum_{a \mid \frac{x_i}{x_k}} \frac{\mu_{S(r)}(a x_k, x_i)}{a^\alpha}$$

$$= \frac{1}{x_k^\alpha} \left(\mu_{S(r)}(x_k, x_i) + \sum_{1 \neq a \mid \frac{x_i}{x_k}} \frac{\mu_{S(r)}(a x_k, x_i)}{a^\alpha} \right).$$

We are going to show that the factor on the right goes to zero for some α.

The proof continues

We have

$$\lim_{\alpha\to\infty}(x_k^\alpha(h_{i,r}(\alpha))=\mu_{S(r)}(x_k,x_i)+\lim_{\alpha\to\infty}\sum_{1\neq a\,|\,\frac{x_i}{x_k}}\underbrace{\frac{\mu_{S(r)}(ax_k,x_i)}{a^\alpha}}_{\to 0\text{ as }\alpha\to\infty}$$

$$=\mu_{S(r)}(x_k,x_i)>0.$$

The definition of the Möbius function $\mu_{S(r)}$ implies that

$$x_k^0(h_{i,r}(0))=\sum_{x_k\,|\,x_j\,|\,x_i}\mu_{S(r)}(x_j,x_i)=\delta_{S(r)}(x_k,x_i)=0,$$

since $x_k\neq x_i$. In addition,

$$\frac{d(x_k^\alpha h_{i,r}(\alpha))}{d\alpha}=\sum_{1\neq a\,|\,\frac{x_i}{x_k}}-\log(a)\frac{\mu_{S(r)}(ax_k,x_i)}{a^\alpha}$$

$$=\left(\sum_{\substack{a\,|\,\frac{x_i}{x_k}\\ a\neq 1,\frac{x_i}{x_k}}}-\log(a)\frac{\mu_{S(r)}(ax_k,x_i)}{a^\alpha}\right)-r\log(p_i)\log\left(\frac{x_i'}{x_k}\right)\frac{\mu_{S(r)}(x_i,x_i)}{\left(\frac{x_i}{x_k}\right)^\alpha}.$$

The proof continues

Thus when the integer r is sufficiently large, we have

$$\frac{d(x_k^\alpha h_{i,r}(\alpha))}{d\alpha}(0) = \sum_{\substack{a \mid \frac{x_i}{x_k} \\ a \neq 1, \frac{x_i}{x_k}}} -\log(a)\mu_{S(r)}(ax_k, x_i)$$

$$- r\log(p_i)\underbrace{\log\left(\frac{x_i'}{x_k}\right)}_{>0}\underbrace{\mu_{S(r)}(x_k, x_i)}_{>0} < 0.$$

Thus the function $x_k^\alpha h_{i,r}(\alpha)$ obtains negative values for some positive α. In addition, $x_k^\alpha h_{i,r}(\alpha)$ is continuous. Now it follows from Bolzano's Theorem that there exists $\alpha_0 \in]0, \infty[$ such that $x_k^{\alpha_0} h_{i,r}(\alpha_0) = 0$ and therefore $h_{i,r}(\alpha_0) = \Psi_{S(r), \frac{1}{N^\alpha}}(x_i) = 0$. Proposition now implies the matrix $[S(r)]_{N^{\alpha_0}}$ has to be singular.

□

A characterization for the class of meet semilattices which yield singular power LCM matrices

Theorem

Let $(L, \leq)$ *be a meet semilattice with n elements, where* $L = \{z_1, z_2, \ldots, z_n\}$. *Then the following conditions are equivalent:*

1. *The LCM matrix* $([x_i, x_j]^\alpha)$ *is nonsingular for all* $\alpha > 0$ *and for all sets* $S = \{x_1, x_2, \ldots, x_n\} \subset \mathbb{Z}^+$ *such that* $(S, |) \cong (L, \leq)$.
2. L *is* $\wedge$*-tree set (i.e. every element covers at most one element).*
3. *For all* $z_i, z_j \in L : (\mu_L(z_i, z_j) > 0 \Rightarrow z_i = z_j)$.

The proof

(1) $\Rightarrow$ (2) First we assume Condition 1. Suppose for a contradiction that some element of L covers more than one element. Suppose that z_i is minimal element such that it covers elements $z_{i_1}, \ldots, z_{i_k} \in L$, where $k \geq 2$. Let $z_r = z_{i_1} \wedge \cdots \wedge z_{i_k}$. If $\mu_L(z_r, z_i) > 0$, then the previous theorem would imply that the matrix $([x_i, x_j]^\alpha)$ is singular for some $\alpha > 0$ and $S \subset \mathbb{Z}_+$, where $(S, |) \cong (L, \leq)$. Thus we must have

$$\mu_L(z_r, z_i) = -\sum_{z_r \leq z_j < z_i} \mu_L(z_r, z_j) \leq 0.$$

Let $z_{l_1}, \ldots, z_{l_m} \in [\![z_r, z_i]\!]$ be the elements that cover z_r. Here $m \geq 2$, since otherwise we would have $z_{l_1} \leq z_{i_1}, \ldots, z_{i_k}$ and further $z_r \prec z_{l_1} \leq z_{i_1} \wedge \cdots \wedge z_{i_k}$. So we know that the terms $\mu_L(z_r, z_{l_1}), \ldots, \mu_L(z_r, z_{l_m})$ appear in the nonnegative sum

$$\begin{aligned} 0 &\leq \sum_{z_r \leq z_j < z_i} \mu_L(z_r, z_j) \\ &= \mu_L(z_r, z_r) + \mu_L(z_r, z_{l_1}) + \cdots + \mu_L(z_r, z_{l_m}) + \sum_{z_{l_1}, \ldots, z_{l_m} < z_j < z_i} \mu_L(z_r, z_j) \\ &= 1 - m + \sum_{z_{l_1}, \ldots, z_{l_m} < z_j < z_i} \mu_L(z_r, z_j). \end{aligned}$$

The proof continues

Therefore there exists $z_j > z_r$ such that $\mu_L(z_r, z_j) > 0$. This means that z_j needs to cover more than one element (otherwise we would have $\mu_L(z_r, z_j) = -1$ or $\mu_L(z_r, z_j) = 0$ by Lemma). This is a contradiction, since z_i was supposed to be minimal element such that it covers at least two elements (and here $z_j < z_i$). Thus condition (2) has to hold.
The implications (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) are straightforward.

$\square$

Fatal consequences of our work

It now follows that also these conjectures by Hong are false:

Conjecture (Hong, J. Algebra, 2004)

Let $\alpha \neq 0$ and let $S = \{x_1, \ldots, x_n\}$ be an odd-gcd-closed set. Then the matrix $[[x_i, x_j]^\alpha]$ is nonsingular.

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Conjecture (Hong, J. Algebra, 2004)

Let $S = \{x_1, \ldots, x_n\}$ be an odd-gcd-closed set and f a completely multiplicative function. If f is strictly monotonous, then the matrix $[f[x_i, x_j]]$ is nonsingular.

Conjecture (Hong, J. Algebra, 2004)

Let $S = \{x_1, \ldots, x_n\}$ be an odd-lcm-closed set and f a completely multiplicative function. If f is strictly monotonous, then the matrix $[f[x_i, x_j]]$ is nonsingular.

Thank you for your attention!!!