Paratransitivity for Algebras of Linear Transformations

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Transitivity

Until further notice, $\mathcal{V}$ will stand for a finite-dimensional $\mathbb{C}$omplex vector space of dimension at least 3, and $\mathcal{L}(\mathcal{V})$ will be the algebra of the linear transformations on $\mathcal{V}$.

Definition

A subalgebra $\mathcal{A}$ of $\mathcal{L}(\mathcal{V})$ is said to be **transitive** if for every non-zero x:

$$\mathcal{A}x = \mathcal{V}.$$

Transitivity

TFAE for a subalgebra $\mathcal{A}$ of $\mathcal{L}(\mathcal{V})$:

1. $\mathcal{A}$ is transitive;
2. $\mathcal{A}$ has no non-trivial (i.e. non-{0}, non-$\mathcal{V}$) invariant subspaces;
3. For every non-zero x and any $y \in \mathcal{V}$, there is some $A \in \mathcal{A}$ such that
$$Ax = y;$$
4. For every pair of one-dimensional subspaces $\mathcal{X}$ and $\mathcal{Y}$, there is some $A \in \mathcal{A}$ such that
$$A\mathcal{X} = \mathcal{Y}; \ \ (\text{equivalently} : A\mathcal{X} \cap \mathcal{Y} \neq \{0\});$$
5. $\mathcal{A} = \mathcal{L}(\mathcal{V})$.

Paratransitivity

Definition

A subalgebra $\mathcal{A}$ of $\mathcal{L}(\mathcal{V})$ is said to be (k, m)**-transitive** if for every k-dimensional subspace $\mathcal{X}$ and every m-dimensional subspace $\mathcal{Y}$ of $\mathcal{V}$:

$$\mathcal{A}\mathcal{X} \cap \mathcal{Y} \neq \{0\}.$$

The notion of $(1, 1)$-transitivity is therefore the usual notion of transitivity for algebras.

First Observations

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3. $\mathcal{A}$ is (k,m)-transitive $\iff$ every k-dimensional subspace $\mathcal{X}$ of $\mathcal{V}$ and every m-dimensional subspace $\mathcal{Y}$ of $\mathcal{V}$: $\langle \mathcal{A}\mathcal{X}, \mathcal{Y} \rangle \neq \{0\}$; ↓[a]

[a] Independent of the choice of the inner product on $\mathcal{V}$.

First Observations (continued); i.e. Second Observations

1. $\mathcal{A}$ is (k, m)-transitive $\Longleftrightarrow$ $\mathcal{A}^{\#}$ is (m, k)-transitive; ↓[a]
2. $\mathcal{A}$ is (k, m)-transitive $\Longleftrightarrow$ $\mathcal{A}^{*}$ is (m, k)-transitive; ↓[b]↓[c]

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[a] $\{\ \}^{\#}$ indicates Banach adjoint.
[b] $\{\ \}^{*}$ indicates Hilbert adjoint.
[c] Independent of the choice of the inner product on $\mathcal{V}$.

Convenient Notation

Notation

When $1 \leq k, p \leq \dim\mathcal{V}$, let us write $\mathcal{A}\langle k\rangle = \langle p\rangle$, to indicate that $\dim(\mathcal{A}\mathcal{W}) = p$ whenever $\mathcal{W}$ is a k-dimensional subspace of $\mathcal{V}$.

The notation such as "$\mathcal{A}\langle k\rangle \leq \langle p\rangle$" and "$\mathcal{A}\langle k\rangle \geq \langle p\rangle$" is now self-explanatory.

Note that $\mathcal{A}$ is (k, m)-transitive if and only if

$$\mathcal{A}\langle k\rangle \geq \langle \dim\mathcal{V} - m + 1\rangle,$$

or equivalently

$$\mathcal{A}\langle k\rangle > \langle \dim\mathcal{V} - m\rangle.$$

“The Usual Suspects”

Theorem

If $\mathcal{W}$, $\mathcal{Z}$ are subspaces of $\mathcal{V}$ such that

1. $\dim(\mathcal{Z}) = k - 1$;
2. $\operatorname{codim}(\mathcal{W}) = m - 1$;

then the algebra $\mathcal{A}$ of all those linear transformations on $\mathcal{V}$ which map $\mathcal{V}$ into $\mathcal{W}$, and vanish on $\mathcal{Z}$, is a (k, m)-transitive subalgebra of $\mathcal{L}(\mathcal{V})$.

Terminology

The algebras of the type described in this theorem shall be denoted by $\mathcal{A}(\mathcal{Z}; \mathcal{W})$ and referred to as “**the usual suspects**”.

Minimality

Definition

A (k, m)-transitive subalgebra $\mathcal{A}$ of $\mathcal{L}(\mathcal{V})$ is **minimal (k, m)-transitive** if it contains no proper (k, m)-transitive subalgebras.

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By the dimensionality considerations, every (k, m)-transitive subalgebra of $\mathcal{L}(\mathcal{V})$ contains a minimal (k, m)-transitive subalgebra of $\mathcal{L}(\mathcal{V})$.

Since (k, m)-transitivity obviously passes from subalgebras to the algebras, minimal (k, m)-transitive subalgebras of $\mathcal{L}(\mathcal{V})$ are the objects of our primary interest.

The Minimal Usual Suspects

Theorem

If $\mathcal{W}$, $\mathcal{Z}$ are subspaces of $\mathcal{V}$ such that

1. $\dim(\mathcal{Z}) = k - 1$;
2. $\operatorname{codim}(\mathcal{W}) = m - 1$;

then $\mathcal{A}(\mathcal{Z}; \mathcal{W})$ is minimal (k, m)-transitive if and only if at least one of the following holds:

- co-$\dim(\mathcal{Z}) = 1$; (i.e. $dim(\mathcal{V}) = k$)
- $\dim(\mathcal{W}) = 1$; (i.e. $dim(\mathcal{V}) = m$)
- $\mathcal{W}$ is not a subspace of $\mathcal{Z}$; (i.e. $\mathcal{A}(\mathcal{Z}; \mathcal{W})$ is not nilpotent of order 2).

$\circ\!\!\neg$: $[\Longrightarrow]$

If $\mathcal{W}$ is a subspace of $\mathcal{Z}$, where $\dim(\mathcal{W}) > 1$, co-$\dim(\mathcal{Z}) > 1$, write

$$\mathcal{Z} = \mathcal{W} \oplus \mathcal{X}, \quad \text{and} \quad \mathcal{V} = \mathcal{Z} \oplus \mathcal{Y}.$$

Let $\mathcal{S}$ be a proper transitive subspace of $\mathcal{L}(\mathcal{Y}, \mathcal{W})$.[2]

Then the proper subalgebra $\begin{bmatrix} 0 & \mathcal{S} \\ 0 & 0 \end{bmatrix}$ of $\mathcal{A}(\mathcal{Z}; \mathcal{W})$ is (k, m)-transitive.

[2] It is known that $\mathcal{L}(\mathcal{Y}, \mathcal{W})$ has transitive subspaces of any dimension greater than or equal to $\dim(\mathcal{Y}) + \dim(\mathcal{W}) - 1$, and so proper such subspaces when $\dim(\mathcal{Y}) \neq 1$ and $\dim(\mathcal{W}) \neq 1$.

Will assume $1 \leq d < dim(\mathcal{V})$

It is easy to see that a subalgebra $\mathcal{A}$ of $\mathcal{L}(\mathcal{V})$ is $(1, dim(\mathcal{V}))$-transitive if and only if it has a trivial common kernel.

From now on we implicitely restrict our attention to the case $1 \leq d < dim(\mathcal{V})$.

We shall also switch between linear transformational and matricial points of view with impunity.

$(1, d)$-Transitivity

The following are equivalent for a subalgebra $\mathcal{A}$ of $\mathcal{L}(\mathcal{V})$:

1. $\mathcal{A}$ is $(1, d)$-transitive;
2. For every $x \neq 0$: co-dim$(\mathcal{A}x) < d$.
3. Every non-trivial invariant subspace of $\mathcal{A}$ has co-dimension less than d; (i.e. $\mathcal{A}$ has no small non-trivial invariant subspaces);
4. For every $x \neq 0$:
   either co-dim$(Rad(\mathcal{A})x) < d$ or co-dim$(WFac(\mathcal{A})x) < d$. $\downarrow$[f]

[f] $WFac(\mathcal{A})$ is an algebra such that $\mathcal{A} = WFac(\mathcal{A}) \dot{+} Rad(\mathcal{A})$

Wedderburn's Principal Theorem

Wedderburn's Principal (Decomposition) Theorem [1908]; (restricted to our setting).

Every subalgebra $\mathcal{A}$ of $\mathcal{L}(\mathcal{V})$ can be written as an internal direct sum of its nil radical and a semi-simple algebra; $\mathcal{A} = \mathcal{S} \dot{+} \mathit{Rad}(\mathcal{A})$.

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Terminology

A decomposition of a subalgebra $\mathcal{A}$ of $\mathcal{L}(\mathcal{V})$ into an internal direct sum $S \dotplus Rad(\mathcal{A})$ is said to be a " **Wedderburn principal decomposition**" of $\mathcal{A}$, and the necessarily semi-simple algebra S is said to be a "**Wedderburn factor**" of $\mathcal{A}$.

Malcev's Addendum

Malcev's addendum [1942]; (restricted to our setting).

If $\mathcal{S}$ and $\mathcal{S}'$ are two Wedderburn factors of $\mathcal{A}$ in $\mathcal{L}(\mathcal{V})$, then $\mathcal{S}' = (I - N)^{-1}\mathcal{S}(I - N)$ for some $N \in Rad(\mathcal{A})$.

Block-Upper-△ Forms

Up to similarity, every subalgebra $\mathcal{A}$ of $\mathbb{M}_n(\mathbb{C})$ has a simultaneous block-upper-△ form, such that each $\mathcal{A}_{ii}$ is either a full matrix algebra or $\{\mathcal{O}_{1\times 1}\}$.

$$\mathcal{A} = \begin{bmatrix} \blacksquare_{11} & \square & \square & \square & \dots & \square \\ 0 & \blacksquare_{22} & \square & \square & \dots & \square \\ 0 & 0 & \blacksquare_{33} & \square & \dots & \square \\ 0 & 0 & 0 & \blacksquare_{44} & \dots & \square \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \blacksquare_{kk} \end{bmatrix}$$

$$\mathcal{A}_{ii} \stackrel{\text{def}}{=} \{\ A_{ii} \mid A \in \mathcal{A}\ \} = \mathbb{M}_{n_i}(\mathbb{C}) \text{ or } \{0_{1\times 1}\}.$$

Block-Upper-△ Forms

Definition

Let us call this **a maximal block-upper-△ form** of $\mathcal{A}$.

$$\mathcal{A} = \begin{bmatrix} \blacksquare_{11} & \square & \square & \square & \dots & \square \\ 0 & \blacksquare_{22} & \square & \square & \dots & \square \\ 0 & 0 & \blacksquare_{33} & \square & \dots & \square \\ 0 & 0 & 0 & \blacksquare_{44} & \dots & \square \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \blacksquare_{kk} \end{bmatrix}$$

$$\mathcal{A}_{ii} \stackrel{\text{def}}{=} \{ \ A_{ii} \mid A \in \mathcal{A} \ \} = \mathbb{M}_{n_i}(\mathbb{C}) \text{ or } \{0_{1\times 1}\}.$$

$\mu(\mathcal{A})$

Notation

A maximal block-upper-△ form may not be unique, but the number *k* of the blocks on the block-diagonal is an intrinsic property of $\mathcal{A}$.
Let us denote this *k* by $\mu(A)$.

$$\begin{bmatrix} \blacksquare_{11} & \square & \square & \square & \dots & \square \\ 0 & \blacksquare_{22} & \square & \square & \dots & \square \\ 0 & 0 & \blacksquare_{33} & \square & \dots & \square \\ 0 & 0 & 0 & \blacksquare_{44} & \dots & \square \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \blacksquare_{\mu(\mathcal{A})\mu(\mathcal{A})} \end{bmatrix}$$

Nil Radical

Since irreducible matrix algebras are simple, when $\mathcal{A}$ is in a maximal block-upper-△ form, the nil radical $Rad(\mathcal{A})$ is exactly the subalgebra of all strictly block-upper-△ matrices in $\mathcal{A}$.

$$Rad(\mathcal{A}) = \begin{bmatrix} 0_{(1)} & \square & \square & \square & \dots & \square \\ 0 & 0_{(2)} & \square & \square & \dots & \square \\ 0 & 0 & 0_{(3)} & \square & \dots & \square \\ 0 & 0 & 0 & 0_{(4)} & \dots & \square \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0_{(\mu(\mathcal{A}))} \end{bmatrix}$$

In particular $\left(Rad(\mathcal{A})\right)^{\mu(\mathcal{A})} = \{0_{n\times n}\}$.

“Unhinged” Block-Upper-△ form

Terminology

A block-upper-△ subalgebra of $\mathbb{M}_n(\mathbb{C})$ is **unhinged** if the algebra is an internal direct sum of its block-diagonal subalgebra and its strictly block-upper-△ subalgebra.

$$\begin{bmatrix} \square_{11} & 0 & 0 & \dots & 0 \\ 0 & \square_{22} & 0 & \dots & 0 \\ 0 & 0 & \square_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \square_{\mu(\mathcal{A})\mu(\mathcal{A})} \end{bmatrix} \dotplus \begin{bmatrix} 0 & \square & \square & \dots & \square \\ 0 & 0 & \square & \dots & \square \\ 0 & 0 & 0 & \dots & \square \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

If $\mathcal{A}$ has an unhinged maximal block-upper-△ form, then it is clear that the block diagonal subalgebra is a Wedderburn factor of $\mathcal{A}$, and the strictly block-upper-△ subalgebra is the nil radical of $\mathcal{A}$.

$$\begin{bmatrix} \blacksquare_{11} & 0 & 0 & \dots & 0 \\ 0 & \blacksquare_{22} & 0 & \dots & 0 \\ 0 & 0 & \blacksquare_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \blacksquare_{\mu(\mathcal{A})\mu(\mathcal{A})} \end{bmatrix} \dot{+} \begin{bmatrix} 0 & \square & \square & \dots & \square \\ 0 & 0 & \square & \dots & \square \\ 0 & 0 & 0 & \dots & \square \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

"A Natural Question"

Question

Given a matrix algebra $\mathcal{A}$ in a maximal block-upper-$\triangle$ form, is it always possible to unhinge $\mathcal{A}$ without "messing up" the underlying spatial decomposition; i.e. via an application of a block-upper-$\triangle$ similarity?

Barker-Eifler-Kezlan Theorem [1978] (conjectured by H. Schneider)

Up to similarity, every subalgebra $\mathcal{A}$ in $\mathbb{M}_n(\mathbb{C})$ has a maximal block-upper-△ form such that *the non-zero block-diagonal positions are either " linked" or "independent"*.

$$\mathcal{A} \sim \left\{ \begin{bmatrix} A & \square & \square & \square & \dots & \square \\ 0 & B & \square & \square & \dots & \square \\ 0 & 0 & 0 & \square & \dots & \square \\ 0 & 0 & 0 & B & \dots & \square \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & A \end{bmatrix} \right\} \left(\supset \left\{ \begin{bmatrix} I & \square & \square & \square & \dots & \square \\ 0 & 0 & \square & \square & \dots & \square \\ 0 & 0 & 0 & \square & \dots & \square \\ 0 & 0 & 0 & 0 & \dots & \square \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & I \end{bmatrix} \right\} \right)$$

Let us call this **a maximal grouped block-upper-△ form** of $\mathcal{A}$.

J. F. Watters [1980]

Given a subalgebra $\mathcal{A}$ of $\mathbb{M}_n(\mathbb{F})$ in a maximal block-upper-$\triangle$ form, after some <u>*block-diagonal similarity*</u> one will arrive at an algebra in a maximal grouped block-upper-$\triangle$ form.

$$\mathcal{A} \sim \left\{ \begin{bmatrix} A & \square & \square & \square & \dots & \square \\ 0 & B & \square & \square & \dots & \square \\ 0 & 0 & 0 & \square & \dots & \square \\ 0 & 0 & 0 & B & \dots & \square \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & A \end{bmatrix} \right\} \left(\supset \left\{ \begin{bmatrix} I & \square & \square & \square & \dots & \square \\ 0 & 0 & \square & \square & \dots & \square \\ 0 & 0 & 0 & \square & \dots & \square \\ 0 & 0 & 0 & 0 & \dots & \square \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & I \end{bmatrix} \right\} \right)$$

Furthermore, *the block-diagonal* of a maximal grouped block-upper-$\triangle$ form is fairly unique: if two such forms are similar, then (up to a block-diagonal similarity) the block-diagonal blocks are simply being permuted.

$$\mathcal{A} \sim \left\{ \begin{bmatrix} A & \square & \square & \square & \dots & \square \\ 0 & B & \square & \square & \dots & \square \\ 0 & 0 & 0 & \square & \dots & \square \\ 0 & 0 & 0 & B & \dots & \square \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & A \end{bmatrix} \right\} \left(\supset \left\{ \begin{bmatrix} I & \square & \square & \square & \dots & \square \\ 0 & 0 & \square & \square & \dots & \square \\ 0 & 0 & 0 & \square & \dots & \square \\ 0 & 0 & 0 & 0 & \dots & \square \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & I \end{bmatrix} \right\} \right)$$

The answer to the "Natural Question" is affirmative when $\mathcal{A}$ is semi-simple.

Theorem

Given a semi-simple subalgebra $\mathcal{A}$ of $\mathbb{M}_n(\mathbb{F})$ in a maximal block-upper-$\triangle$ form, there is a *block-upper-$\triangle$ similarity* which implements the compression to the block-diagonal of $\mathcal{A}$;

i.e. there is an invertible *block-upper-$\triangle$* T such that

$$T^{-1}AT = BlockDiag(A), \quad \text{for all} \quad A \in \mathcal{A}$$

Spatial Wedderburn Decomposition of Semi-Simple Matrix Algebras

Of course, via Barker-Eifler-Kezlan-Watters' theorem, we can insure that the block-diagonal algebra $\mathit{BlockDiag}(\mathcal{A})$ thus obtained has a particularly simple grouped form.

$$\left\{\begin{bmatrix} A & 0 & 0 & 0 & \dots & 0 \\ 0 & B & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & B & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & A \end{bmatrix}\right\} \left(\supset \left\{\begin{bmatrix} I & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & I \end{bmatrix}\right\} \right)$$

At this point we can also use a block-permutation similarity to "shuffle" the block-diagonal algebra so that the "linked" blocks become adjacent.

Spatial Wedderburn Decomposition of Semi-Simple Matrix Algebras

We have arrived at the following conclusion:

If $\mathcal{A}$ is a semi-simple subalgebra of $\mathbb{M}_n(\mathbb{C})$, then $\mathcal{A}$ is **simulatneously similar** to an internal direct sum

$$\bigoplus_i \left(\mathbb{M}_{k_i}(\mathbb{C}) \otimes I_{p_i}\right)$$

Spatial Wedderburn Decomposition of Semi-Simple Matrix Algebras

In fact more is true:

If $\mathcal{A}$ is a semi-simple subalgebra of $\mathbb{M}_n(\mathbb{F})$, then there exist irreducible division algebras $\mathcal{D}_i$ of matrices over $\mathbb{F}$, such that $\mathcal{A}$ is **simulatneously similar** to an internal direct sum

$$\bigoplus_i \left(\mathbb{M}_{k_i}(\mathcal{D}_i) \otimes I_{p_i}\right)$$

Spatial Wedderburn Decomposition of Semi-Simple Matrix Algebras

If $\mathcal{A}$ is a semi-simple subalgebra of $\mathbb{M}_n(\mathbb{F})$, then there exist irreducible division algebras $\mathcal{D}_i$ of matrices over $\mathbb{F}$, such that $\mathcal{A}$ is **simulatneously similar** to an internal direct sum

$$\bigoplus_i \left(\mathbb{M}_{k_i}(\mathcal{D}_i) \otimes I_{p_i}\right)$$

Compare this to the classical theorem of Wedderburn which states that a semi-simple finite-dimensional $\mathbb{F}$-algebra $\mathcal{A}$ can be written, up to an algebra isomorphism (and quite uniquely), as a direct sum of full matrix algebras over division $\mathbb{F}$-algebras.

Affirmative General Answer to the "Natural Question"

Theorem

Given a subalgebra $\mathcal{A}$ of $\mathbb{M}_n(\mathbb{C})$ in a maximal block-upper-$\triangle$ form, after some *block-upper-$\triangle$ similarity* one will arrive at an algebra in an **unhinged maximal grouped block-upper-$\triangle$ form.**

Affirmative General Answer to the "Natural Question"

Theorem

Given a subalgebra $\mathcal{A}$ of $\mathbb{M}_n(\mathbb{C})$ in a maximal block-upper-$\triangle$ form, after some *block-upper-$\triangle$ similarity* one will arrive at an algebra in an **unhinged maximal grouped block-upper-$\triangle$ form.**

Corollary

If $\mathcal{S}$ is any Wedderburn factor of a subalgebra $\mathcal{A}$ of $\mathbb{M}_n(\mathbb{C})$, then after an appropriate change of basis, $\mathcal{A}$ sports an unhinged maximal grouped block-upper-$\triangle$ form, where the elements of $\mathcal{S}$ are exactly the block-diagonal matrices.

These results are also true for algebras in a general field setting (but there are some restrictions).

A Neat Application

Recall that $\mu(A)$ stands for the number of blocks on the block-diagonal of any maximal block-upper-$\triangle$ form of a matrix algebra $\mathcal{A}$, and that $\left(Rad(\mathcal{A})\right)^{\mu(\mathcal{A})} = \{0\}$.

Theorem

If $\mathbb{F}$ is *algebraically closed*, the following are equivalent for a subalgebra $\mathcal{A}$ of $\mathbb{M}_n(\mathbb{F})$:

1. $\left(Rad(\mathcal{A})\right)^{\mu(\mathcal{A})-1} \neq \{0\}$;
2. There is an element $R \in Rad(\mathcal{A})$ such that $R^{\mu(\mathcal{A})-1} \neq 0$.

In such a case, $\mathcal{A}$ is unicellular, i.e. the lattice of the invariant subspaces of $\mathcal{A}$ is totally ordered by inclusion.

It is certainly known to the algebraists that the use of $\mu(\mathcal{A})$ is essential in the result above. For example, consider the nilpotent algebra

$$\mathcal{A} = \left\{ \begin{bmatrix} 0 & a & b & c \\ 0 & 0 & 0 & -b \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{bmatrix} \middle| \ a, b, c \in \mathbb{C} \right\}.$$

It is clear that $\mathcal{A}^3 = \{0\}$, but $\mathcal{A}^2 \neq \{0\}$ since

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \circ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \neq 0.$$

Yet it is easy to check that $T^2 = 0$ for every $T \in \mathcal{A}$.

Essential Lemma 1 (Back to $(1, d)$-Transitivity ...)

Lemma

If a minimal $(1, d)$-transitive subalgebra $\mathcal{A}$ of $\mathcal{L}(\mathcal{V})$ has a maximal block-upper-$\triangle$ form

$$\begin{bmatrix} \blacksquare_{11} & \square & \square & \square & \dots & \square \\ 0 & \blacksquare_{22} & \square & \square & \dots & \square \\ 0 & 0 & \blacksquare_{33} & \square & \dots & \square \\ 0 & 0 & 0 & \blacksquare_{44} & \dots & \square \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \blacksquare_{\mu(\mathcal{A})\mu(\mathcal{A})} \end{bmatrix}$$

then

$$n_i \leq n - d + 1, \qquad \text{for all} \quad i.$$

Essential Lemma 1 (a different formulation)

Lemma

If a subalgebra $\mathcal{A}$ of $\mathcal{L}(\mathcal{V})$, minimal with respect to the property $\mathcal{A}\langle 1 \rangle \geq \langle p \rangle$, is expressed in a maximal block-upper-$\triangle$ form

$$\begin{bmatrix} \blacksquare_{11} & \square & \square & \square & \dots & \square \\ 0 & \blacksquare_{22} & \square & \square & \dots & \square \\ 0 & 0 & \blacksquare_{33} & \square & \dots & \square \\ 0 & 0 & 0 & \blacksquare_{44} & \dots & \square \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \blacksquare_{\mu(\mathcal{A})\mu(\mathcal{A})} \end{bmatrix}$$

then

$$n_i \leq p, \qquad \text{for all} \quad i.$$

Essential Lemma 2

Lemma

If $\mathcal{A}$ is a minimal $(1, d)$-transitive subalgebra of $\mathcal{L}(\mathcal{V})$, then the invariant subspaces of co-dimension $d - 1$ are exactly the minimal non-trivial invariant subspaces of $\mathcal{A}$.

Consequently, there is an x such that

$$\text{co-dim}\big(\mathcal{A}x\big) = d - 1.$$

Essential Lemma 2 (alternate formulation)

Lemma

If a subalgebra $\mathcal{A}$ of $\mathcal{L}(\mathcal{V})$ is minimal with respect to the property $\mathcal{A}\langle 1\rangle \geq \langle p\rangle$, then the invariant subspaces of dimension p are exactly the minimal non-trivial invariant subspaces of $\mathcal{A}$.

Consequently, there is an x such that

$$dim(\mathcal{A}x) = p.$$

Semi-Simple $(1, d)$-Transitive Matrix Algebras

If a subalgebra $\mathcal{A}$ of $\mathbb{M}_n(\mathbb{C})$ has a maximal block-diagonal form

$$\begin{bmatrix} \blacksquare_{11} & 0 & 0 & 0 & \dots & 0 \\ 0 & \blacksquare_{22} & 0 & 0 & \dots & 0 \\ 0 & 0 & \blacksquare_{33} & 0 & \dots & 0 \\ 0 & 0 & 0 & \blacksquare_{44} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \blacksquare_{kk} \end{bmatrix},$$

then

$$\mathcal{A} \text{ is } (1, d)\text{-transitive } \textbf{if and only if } n_i > n - d, \quad \text{for all} \quad i.$$

Semi-Simple $(1, d)$-Transitive Matrix Algebras

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$$\begin{bmatrix} \blacksquare_{11} & 0 & 0 & 0 & \dots & 0 \\ 0 & \blacksquare_{22} & 0 & 0 & \dots & 0 \\ 0 & 0 & \blacksquare_{33} & 0 & \dots & 0 \\ 0 & 0 & 0 & \blacksquare_{44} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \blacksquare_{kk} \end{bmatrix},$$

then

$$\mathcal{A}\langle 1\rangle \geq \langle p\rangle \textbf{ if and only if } n_i \geq p, \quad \text{for all} \quad i.$$

Semi-Simple Minimal $(1, d)$-Transitive Algebras

Theorem

A subalgebra $\mathcal{A}$ of $\mathbb{M}_n(\mathbb{C})$ in a maximal grouped block-diagonal form is minimal $(1, d)$-transitive [a] **if and only if** $d = n - \frac{n}{k} + 1$ and

$$\mathcal{A} = \left\{ \begin{bmatrix} A & 0 & 0 & 0 & \dots & 0 \\ 0 & A & 0 & 0 & \dots & 0 \\ 0 & 0 & A & 0 & \dots & 0 \\ 0 & 0 & 0 & A & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & A \end{bmatrix} \;\middle|\; A \in \mathbb{M}_{\frac{n}{k}}(\mathbb{C}) \right\} = \mathbb{M}_{\frac{n}{k}}(\mathbb{C}) \otimes I_k$$

[a] $1 \leq d < n$

Semi-Simple Minimal $(1, d)$-Transitive Algebras

Theorem

A subalgebra $\mathcal{A}$ of $\mathbb{M}_n(\mathbb{C})$ in a maximal grouped block-diagonal form is minimal with respect to the property $\mathcal{A}\langle 1\rangle \geq \langle p\rangle$ **if and only if** $(1 \neq)\ p|n$ and

$$\mathcal{A} = \left\{ \begin{bmatrix} A & 0 & 0 & 0 & \dots & 0 \\ 0 & A & 0 & 0 & \dots & 0 \\ 0 & 0 & A & 0 & \dots & 0 \\ 0 & 0 & 0 & A & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & A \end{bmatrix} \middle| \ A \in \mathbb{M}_p(\mathbb{C}) \right\} = \mathbb{M}_p(\mathbb{C}) \otimes I_{n/p}$$

i.e.

A minimal $(1, d)$-transitive subalgebra $\mathcal{A}$ of $\mathbb{M}_n(\mathbb{C})$ $\downarrow$[a] is semi-simple **if and only if** $d = n - \frac{n}{k} + 1$ and $\mathcal{A}$ is simultaneously similar to $\mathbb{M}_{\frac{n}{k}}(\mathbb{C}) \otimes I_k$.

[a] $1 \leq d < n$

i.e.

A semi-simple subalgebra $\mathcal{A}$ of $\mathbb{M}_n(\mathbb{C})$ is minimal with respect to the property $\mathcal{A}\langle 1\rangle \geq \langle p\rangle$ **if and only if** $(1 \neq)\ p|n$ and $\mathcal{A}$ is simultaneously similar to $\mathbb{M}_p(\mathbb{C}) \otimes I_{n/p}$

Canonical Minimal $(1, d)$-Transitive Types

Recall that we have seen one canonical type of a minimal $(1, d)$-transitive matrix algebra so far:

"The Usual Suspects" type (not semi-simple)

$$\mathcal{A} = \left\{ \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \;\middle|\; A \in \mathbb{M}_{n-(d-1)}(\mathbb{C});\; B \in \mathbb{M}_{(n-(d-1))\times(d-1)}(\mathbb{C}) \right\} \subset \mathbb{M}_n(\mathbb{C})$$

Of course these algebras are in a maximal block-upper-$\triangle$ form, and have a non-trivial nil radical.

Canonical Minimal $(1, d)$-Transitive Types

"The Usual Suspects" type (not semi-simple)

$$\mathcal{A} = \left\{ \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \;\middle|\; A \in \mathbb{M}_{n-(d-1)}(\mathbb{C});\; B \in \mathbb{M}_{(n-(d-1))\times(d-1)}(\mathbb{C}) \right\} \subset \mathbb{M}_n(\mathbb{C})$$

Now we also have a canonical semi-simple type:

"The New Kids on the Block (-Diagonal)" type (semi-simple)

$d = 1 \quad : \quad \mathcal{A} = \mathbb{M}_n(\mathbb{C}) \qquad$ ("smallest size"; Burnside's)

$d = \frac{n}{2} + 1 \quad : \quad \mathcal{A} = \mathbb{M}_{\frac{n}{2}}(\mathbb{C}) \otimes I_2 \qquad$ ("just over half the size")

$d = \frac{2n}{3} + 1 \quad : \quad \mathcal{A} = \mathbb{M}_{\frac{n}{3}}(\mathbb{C}) \otimes I_3 \qquad$ ("just over two-thirds the size")

etc.

"Para-Burnside's" For "Smaller" d's.

Theorem

If $d \leq \lceil \frac{n}{2} \rceil$ then the minimal $(1, d)$-transitive matrix algebras are exactly "the usual suspects".

Theorem

If $d = \frac{n}{2} + 1$, then then the minimal $(1, d)$-transitive matrix algebras are exactly "the usual suspects" and "the new kids on the block": $\left(\mathbb{M}_{\frac{n}{2}}(\mathbb{C}) \otimes I_2\right)$.

“Para-Burnside’s” For “Smaller” d’s.

Theorem

If $p > \lfloor \frac{n}{2} \rfloor$ then the algebras minimal with respect to the property $\mathcal{A}\langle 1 \rangle \geq \langle p \rangle$ are exactly “the usual suspects”.

Theorem

If $p = \frac{n}{2}$ then the algebras minimal with respect to the property $\mathcal{A}\langle 1 \rangle \geq \langle p \rangle$ are exactly “the usual suspects” and “the new kids on the block”: $\left(\mathbb{M}_p(\mathbb{C}) \otimes I_2\right)$.

Still open:

What are the minimal $(1, d)$-transitive algebras for slightly bigger d’s?

Basic Observations

The following are equivalent for a subalgebra $\mathcal{A}$ of $\mathcal{L}(\mathcal{V})$ and $d \leq dim(\mathcal{V})$:

1. $\mathcal{A}$ is $(2, d)$-transitive;
2. For every linearly independent pair x, y: co-dim$(\mathcal{A}x + \mathcal{A}y) < d$;
3. Every non-trivial invariant subspace of $\mathcal{A}$ is either one-dimensional or has co-dimension less than d.

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Of course, every $(1, d)$-transitive algebra is automatically $(2, d)$-transitive.

If $d = dim(\mathcal{V})$ then $\mathcal{A}$ is $(2, d)$-transitive if and only if the common kernel of $\mathcal{A}$ is at most 1-dimensional. So, again, **we shall only consider the case $d \leq dim(\mathcal{V}) - 1$ henceforth.**

An Essential Lemma

Unfortunately the direct analogues of the Essential Lemmas for $(1, d)$-transitivity do not hold true for $(2, d)$-transitivity.

Fortunately there is still something we can say in this case.

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Essential Lemma

If $d < dim(\mathcal{V}) - 1$ then every $(2, d)$-transitive subalgebra of $\mathcal{L}(\mathcal{V})$ has at most one 1-dimensional invariant subspace.

If $d \leq \left\lceil \frac{\dim(\mathcal{V})}{2} \right\rceil$, then every minimal $(2, d)$-transitive subalgebra $\mathcal{A}$ of $\mathcal{L}(\mathcal{V})$ has exactly one 1-dimensional invariant subspace.

An Essential Lemma

Unfortunately the direct analogues of the Essential Lemmas for $(1, d)$-transitivity do not hold true for $(2, d)$-transitivity.

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Essential Lemma

If $d < dim(\mathcal{V}) - 1$ then every $(2, d)$-transitive subalgebra of $\mathcal{L}(\mathcal{V})$ has at most one 1-dimensional invariant subspace.

If $d \leq \left\lceil \frac{\dim(\mathcal{V})}{2} \right\rceil$, then every minimal $(2, d)$-transitive subalgebra $\mathcal{A}$ of $\mathcal{L}(\mathcal{V})$ has exactly one 1-dimensional invariant subspace.

i.e. if $p > \left\lfloor \frac{\dim(\mathcal{V})}{2} \right\rfloor$ then minimal $\mathcal{A}\langle 2 \rangle \geq \langle p \rangle$ algebras have exactly one 1-dimensional invariant subspace.

"Para-Burnside's" for Smaller d's and Larger p's

Theorem

If $d \leq \left\lfloor \frac{\dim(\mathcal{V})}{2} \right\rfloor$, then the minimal $(2, d)$-transitive subalgebras of $\mathcal{L}(\mathcal{V})$ are exactly "the usual suspects".

"Para-Burnside's" for Smaller d's and Larger p's

Theorem

If $d \leq \left\lfloor \frac{\dim(\mathcal{V})}{2} \right\rfloor$, then the minimal $(2, d)$-transitive subalgebras of $\mathcal{L}(\mathcal{V})$ are exactly "the usual suspects".

In other words, for $p > \left\lceil \frac{\dim(\mathcal{V})}{2} \right\rceil$, the minimal $\mathcal{A}\langle 2 \rangle \geq \langle p \rangle$ subalgebras of $\mathcal{L}(\mathcal{V})$ are exactly the usual suspects.

Semi-Simple $(2, d)$-Transitive Algebras

Lemma (LL)

The following are equivalent for a semi-simple subalgebra $\mathcal{A}$ of $\mathbb{M}_n(\mathbb{C})$:

1. $\mathcal{A}$ is $(2, d)$-transitive, but not $(1, d)$-transitive;
2.
   1. Case "$d < n - 1$": $\mathcal{A}$ is simultaneously similar to either $\mathcal{B} \oplus \{0_{1\times 1}\}$ or to $\mathcal{B} \oplus \mathbb{C} I_1$, where $\mathcal{B}$ is a $(1, d-1)$-transitive semi-simple subalgebra of $\mathbb{M}_{n-1}(\mathbb{C})$; (i.e. $\mathcal{B}\langle 1\rangle \geq \langle n - d + 1\rangle$) .
   2. Case "$d = n - 1$": $\mathcal{A}$ is simultaneously similar to either $\mathcal{B} \oplus \{0_{1\times 1}\}$ or to $\mathcal{B} \oplus \mathbb{C} I_{n_2} \oplus \ldots \oplus \mathbb{C} I_{n_k}$ $(k \geq 2)$, where $\mathcal{B}$ is a (semi-simple) algebra such that $\mathcal{B}\langle 1\rangle \geq \langle 2\rangle$, and it may be absent in the second case.

Semi-Simple Minimal $(2, d)$-Transitive Algebras

Theorem (LL)

Up to similarity, the full list of the minimal $(2, d)$-transitive subalgebras $\downarrow$[a] that are semi-simple, is this:

			minimal with respect to:
1.	$\mathbb{C} I_n$		$\mathcal{A}\langle 2 \rangle \geq \langle 2 \rangle$;
4.	$\mathbb{M}_m(\mathbb{C}) \otimes I_k$	$2 < m$ & $1 < k$	$\mathcal{A}\langle 2 \rangle \geq \langle m \rangle$;
5.	$\left(\mathbb{M}_m(\mathbb{C}) \otimes I_k\right) \oplus \{0\}$	$2 \leq m$	$\mathcal{A}\langle 2 \rangle \geq \langle m \rangle$;

[a] $1 \leq d < n =$ the dimension of the underlying space, which is at least 3.

Still Open

What are the minimal $(2, d)$-transitive algebras for slightly bigger d's (slightly smaller p's)?

Exactly (k,m)-Transitive Algebras

For the rest of the talk, we shall include the case $dim\mathcal{V} = 2$ in the consideration.

We can completely classify the subalgebras $\mathcal{A}$ of $\mathcal{L}(\mathcal{V})$ such that:

1. $\mathcal{A}\langle 1\rangle = \langle m\rangle$;

Exactly (k,m)-Transitive Algebras

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Exactly (k,m)-Transitive Algebras

For the rest of the talk, we shall include the case $dim\mathcal{V} = 2$ in the consideration.

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2. $\mathcal{A}$ is minimal with respect to the property $\mathcal{A}\langle 2\rangle = \langle m\rangle$;
3. $\mathcal{A}$ is maximal with respect to the property $\mathcal{A}\langle k\rangle = \langle m\rangle$;

Exactly (k,m)-Transitive Algebras

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1. $\mathcal{A}\langle 1\rangle = \langle m\rangle$;
2. $\mathcal{A}$ is minimal with respect to the property $\mathcal{A}\langle 2\rangle = \langle m\rangle$;
3. $\mathcal{A}$ is maximal with respect to the property $\mathcal{A}\langle k\rangle = \langle m\rangle$;
4. $\mathcal{A}$ has the property $\mathcal{A}\langle k\rangle \leq \langle m\rangle$, where $k \geq m$.

“The Exactly Usual Suspects”

Theorem

If $\mathcal{W}$, $\mathcal{Z}$ are subspaces of $\mathcal{V}$ such that

1. $\dim(\mathcal{Z}) = k - 1$;
2. $\dim(\mathcal{W}) = m$;

then the algebra $\mathcal{A}$ of all those linear transformations on $\mathcal{V}$ which map $\mathcal{V}$ into $\mathcal{W}$, and vanish on $\mathcal{Z}$, satisfies the condition

$$\mathcal{A}\langle k \rangle = \langle m \rangle.$$

Terminology

The algebras of the type described in this theorem shall be denoted by $\mathcal{A}_e(\mathcal{Z}; \mathcal{W})$ and referred to as “**the exactly usual suspects**”.

The Minimal Exactly Usual Suspects

Theorem

If $\mathcal{W}$, $\mathcal{Z}$ are subspaces of $\mathcal{V}$ such that

1. $\dim(\mathcal{Z}) = k - 1$;
2. $\dim(\mathcal{W}) = m$;

then $\mathcal{A}_e(\mathcal{Z}; \mathcal{W})$ is minimal such that $\mathcal{A}\langle k\rangle = \langle m\rangle$ if and only if at least one of the following holds:

- $\dim(\mathcal{W}) = 1$; (i.e. $m = 1$)
- co-dim(Z) = 1 (i.e. k = dim(V))
- $\mathcal{W}$ is not a subspace of $\mathcal{Z}$; (i.e. $\mathcal{A}_e(\mathcal{Z}; \mathcal{W})$ is not nilpotent of order 2).

Obviously the possibility of non-minimality only comes into play when $k > m > 1$.

$\mathcal{A}\langle 1\rangle = \langle m\rangle$

Theorem

Subalgebras $\mathcal{A}$ of $\mathcal{L}(\mathcal{V})$ that satisfy

$$\mathcal{A}\langle 1\rangle = \langle m\rangle$$

are the exactly usual suspects,

except in the case $m = 1$, where the algebra $\mathbb{C}I_{\mathcal{V}}$ should be added to the list.

Minimal $\mathcal{A}\langle 2\rangle = \langle m\rangle$, for $m \neq 2$

Theorem

For $m \neq 2$, the minimal $\mathcal{A}\langle 2\rangle = \langle m\rangle$ subalgebras of $\mathcal{L}(\mathcal{V})$ are (up to similarity, of course) exactly the exactly usual suspects.

Minimal $\mathcal{A}\langle 2\rangle = \langle 2\rangle$

Theorem

A subalgebra $\mathcal{A}$ of $\mathcal{L}(\mathcal{V})$ is minimal with respect to the condition $\mathcal{A}\langle 2\rangle = \langle 2\rangle$ if and only if it is (up to similarity, of course) one of the following:

1. an exactly usual suspect;
2. $\mathbb{C}I_{\mathcal{V}}$;
3. $$\mathcal{A} = \left\{ \begin{bmatrix} \alpha I_{\mathcal{W}} & T \\ 0 & 0 \end{bmatrix} \middle| \ \alpha \in \mathbb{C}, \ \ T \in \mathcal{M} \right\},$$
   where $\dim \mathcal{V} > 2 = \dim \mathcal{W}$.
   Here $\mathcal{A}$ is being expressed with respect to a decomposition $\mathcal{V} = \mathcal{W} \dotplus \mathcal{Z}$, and $\mathcal{M}$ is a minimal transitive subspace of $\mathcal{L}(\mathcal{Z}, \mathcal{W})$.

Maximal $\mathcal{A}\langle k\rangle = \langle m\rangle$

Theorem

Subalgebras $\mathcal{A}$ of $\mathcal{L}(\mathcal{V})$ that are maximal with respect to the property

$$\mathcal{A}\langle k\rangle = \langle m\rangle$$

are exactly the algebras $\mathcal{L}(\mathcal{V}, \mathcal{W})$, where $\mathcal{W}$ is an m-dimensional subspace of $\mathcal{V}$,

except in the case $m = k$, where the algebra $\mathbb{C}I_{\mathcal{V}}$ should be added to the list.

$\mathcal{A}\langle k \rangle \leq \langle m \rangle$, where $k \geq m$

Theorem

Given $k \geq m$, the subalgebras $\mathcal{A}$ of $\mathcal{L}(\mathcal{V})$ that satisfy

$$\mathcal{A}\langle k \rangle \leq \langle m \rangle,$$

are exactly the subalgebras of the algebras $\mathcal{L}(\mathcal{V}, \mathcal{W})$ where $\mathcal{W}$ is an m-dimensional subspace of $\mathcal{V}$,

except in the case $m = k$ where the algebras $\mathbb{C}E$ should be added to the list, for all non-zero idempotents E.

Thanks for your attention!

Paratransitivity for Algebras of Linear Transformations

Presented by Leo Livshits[3]

Colby College, Maine, USA

June 9, 2014

LAW 2014; Ljubljana, Slovenia

[3] **Joint work with Gordon MacDonald, Laurent Marcoux and Heydar Radjavi**