Paratransitivity for Algebras of Linear Transformations

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June 9, 2014 LAW 2014; Ljubljana, Slovenia

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Transitivity

Until further notice, \mathcal{V} will stand for a finite-dimensional Complex vector space of dimension at least 3, and $\mathcal{L}(\mathcal{V})$ will be the algebra of the linear transformations on \mathcal{V} .

Definition

A subalgebra \mathcal{A} of $\mathcal{L}(\mathcal{V})$ is said to be **transitive** if for every non-zero *x*:

$$\mathcal{A} x = \mathcal{V}.$$

Transitivity

TFAE for a subalgebra \mathcal{A} of $\mathcal{L}(\mathcal{V})$:

- \mathcal{A} is transitive;
- 2 \mathcal{A} has no non-trivial (i.e. non-{0}, non- \mathcal{V}) invariant subspaces;
- So For every non-zero x and any $y \in \mathcal{V}$, there is some $A \in \mathcal{A}$ such that

$$Ax = y;$$

● For every pair of one-dimensional subspaces X and Y, there is some A ∈ A such that

$$A\mathcal{X} = \mathcal{Y}; \ (equivalently : A\mathcal{X} \cap \mathcal{Y} \neq \{0\});$$

Paratransitivity

Definition

A subalgebra \mathcal{A} of $\mathcal{L}(\mathcal{V})$ is said to be (k, m)-transitive if

for every k-dimensional subspace \mathcal{X} and

every *m*-dimensional subspace \mathcal{Y} of \mathcal{V} :

 $\mathcal{AX} \cap \mathcal{Y} \neq \{0\}.$

The notion of (1, 1)-transitivity is therefore the usual notion of transitivity for algebras.

First Observations

 A is (k, m)-transitive ⇐⇒ A maps subspaces of dimension k or larger to subspaces of co-dimension smaller than m;

First Observations

- A is (k, m)-transitive ⇔ for every k-dimensional subspace X of V and every m-dimensional subspace Y of V[#]: [AX, Y] ≠ {0};

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- A is (k, m)-transitive ⇔ for every k-dimensional subspace X of V and every m-dimensional subspace Y of V[#]: [AX, Y] ≠ {0};
- Is (k, m)-transitive ⇒ every k-dimensional subspace X of V and every m-dimensional subspace Y of V: ⟨AX, Y⟩ ≠ {0}; ↓^a

^{*a*} Independent of the choice of the inner product on \mathcal{V} .

First Observations (continued);

i.e. Second Observations

)
$$\mathcal{A}$$
 is (k, m) -transitive $\iff \mathcal{A}^{\#}$ is (m, k) -transitive; \downarrow^{a}

2 \mathcal{A} is (k, m)-transitive $\iff \mathcal{A}^*$ is (m, k)-transitive; $\downarrow b \downarrow c$

- ^{*a*} { }[#] indicates Banach adjoint.
- ^b { }* indicates Hilbert adjoint.

^c Independent of the choice of the inner product on \mathcal{V} .

Convenient Notation

Notation

When $1 \le k, p \le \dim \mathcal{V}$, let us write $\mathcal{A}\langle k \rangle = \langle p \rangle$, to indicate that $\dim(\mathcal{AW}) = p$ whenever \mathcal{W} is a *k*-dimensional subspace of \mathcal{V} .

The notation such as " $A\langle k \rangle \leq \langle p \rangle$ " and " $A\langle k \rangle \geq \langle p \rangle$ " is now self-explanatory.

Note that A is (k, m)-transitive if and only if

$$\mathcal{A}\langle k \rangle \geq \langle \dim \mathcal{V} - m + 1 \rangle$$
,

or equivalently

 $\mathcal{A}\langle \mathbf{k}\rangle > \langle \dim \mathcal{V} - \mathbf{m} \rangle.$

"The Usual Suspects"

Theorem

If \mathcal{W}, \mathcal{Z} are subspaces of \mathcal{V} such that

$$lim(\mathcal{Z}) = k - 1;$$

2
$$\operatorname{codim}(\mathcal{W}) = m - 1;$$

then the algebra \mathcal{A} of all those linear transformations on \mathcal{V} which map \mathcal{V} into \mathcal{W} , and vanish on \mathcal{Z} , is a (k, m)-transitive subalgebra of $\mathcal{L}(\mathcal{V})$.

Terminology

The algebras of the type described in this theorem shall be denoted by $\mathcal{A}(\mathcal{Z}; \mathcal{W})$ and referred to as "the usual suspects".

Minimality

Definition

A (k, m)-transitive subalgebra \mathcal{A} of $\mathcal{L}(\mathcal{V})$ is minimal (k, m)-transitive if it contains no proper (k, m)-transitive subalgebras.

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Definition

A (k, m)-transitive subalgebra \mathcal{A} of $\mathcal{L}(\mathcal{V})$ is minimal (k, m)-transitive if it contains no proper (k, m)-transitive subalgebras.

By the dimensionality considerations, every (k, m)-transitive subalgebra of $\mathcal{L}(\mathcal{V})$ contains a minimal (k, m)-transitive subalgebra of $\mathcal{L}(\mathcal{V})$.

Since (k, m)-transitivity obviously passes from subalgebras to the algebras, minimal (k, m)-transitive subalgebras of $\mathcal{L}(\mathcal{V})$ are the objects of our primary interest.

The Minimal Usual Suspects

Theorem

If $\mathcal W,\,\mathcal Z$ are subspaces of $\mathcal V$ such that

$$fin(\mathcal{Z}) = k - 1;$$

2
$$\operatorname{codim}(\mathcal{W}) = m - 1;$$

then $\mathcal{A}(\mathcal{Z}; \mathcal{W})$ is <u>minimal</u> (*k*, *m*)-transitive if and only if at least one of the following holds:

- co-dim $(\mathcal{Z}) = 1$; (i.e. $dim(\mathcal{V}) = k$)
- dim(W) = 1; (i.e. dim(V) = m)
- \mathcal{W} is not a subspace of \mathcal{Z} ; (i.e. $\mathcal{A}(\mathcal{Z}; \mathcal{W})$ is not nilpotent of order 2).

○¬: [⇒]

If \mathcal{W} is a subspace of \mathcal{Z} , where dim $(\mathcal{W}) > 1$, co-dim $(\mathcal{Z}) > 1$, write

$$\mathcal{Z} = \mathcal{W} \oplus \mathcal{X}$$
, and $\mathcal{V} = \mathcal{Z} \oplus \mathcal{Y}$.

Let S be a proper transitive subspace of $\mathcal{L}(\mathcal{Y}, \mathcal{W})$.²

Then the proper subalgebra
$$\begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix}$$
 of $\mathcal{A}(\mathcal{Z}; \mathcal{W})$ is (k, m) -transitive.

² It is known that $\mathcal{L}(\mathcal{Y}, \mathcal{W})$ has transitive subspaces of any dimension greater than or equal to $\dim(\mathcal{Y}) + \dim(\mathcal{W}) - 1$, and so proper such subspaces when $\dim(\mathcal{Y}) \neq 1$ and $\dim(\mathcal{W}) \neq 1$.

Will assume $1 \le d < dim(\mathcal{V})$

It is easy to see that a subalgebra \mathcal{A} of $\mathcal{L}(\mathcal{V})$ is $(1, dim(\mathcal{V}))$ -transitive if and only if it has a trivial common kernel.

From now on we implicitely restrict our attention to the case $1 \le d < dim(\mathcal{V})$.

We shall also switch between linear transformational and matricial points of view with impunity.

(1, *d*)-Transitivity

The following are equivalent for a subalgebra \mathcal{A} of $\mathcal{L}(\mathcal{V})$:

- \mathcal{A} is (1, *d*)-transitive;
- **2** For every $x \neq 0$: co-dim(Ax) < d.
- Every non-trivial invariant subspace of A has co-dimension less than d; (i.e. A has no small non-trivial invariant subspaces);
- For every $x \neq 0$: either co-dim(Rad(A)x) < d or co-dim(WFac(A)x) < d. \downarrow^{f}

^{*f*} WFac(A) is an algebra such that A = WFac(A) + Rad(A)

Wedderburn's Principal Theorem

Wedderburn's Principal (Decomposition) Theorem [1908]; (restricted to our setting).

Every subalgebra $\mathcal A$ of $\mathcal L(\mathcal V)$ can be written as an internal direct sum of

its nil radical and a semi-simple algebra; $\mathcal{A} = \mathcal{S} + Rad(\mathcal{A})$.

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Terminology

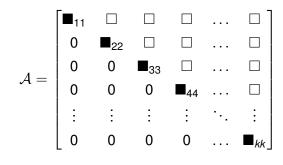
A decomposition of a subalgebra \mathcal{A} of $\mathcal{L}(\mathcal{V})$ into an internal direct sum $S \dotplus Rad(\mathcal{A})$ is said to be a "Wedderburn principal decomposition" of \mathcal{A} , and the necessarily semi-simple algebra S is said to be a "Wedderburn factor" of \mathcal{A} .

Malcev's Addendum

Malcev's addendum [1942]; (restricted to our setting). If S and S' are two Wedderburn factors of A in $\mathcal{L}(V)$, then $S' = (I - N)^{-1}S(I - N)$ for some $N \in Rad(A)$.

Block-Upper- \triangle Forms

Up to similarity, every subalgebra \mathcal{A} of $\mathbb{M}_n(\mathbb{C})$ has a simultaneous block-upper- \triangle form, such that each \mathcal{A}_{ii} is either a full matrix algebra or $\{\mathcal{O}_{1\times 1}\}.$

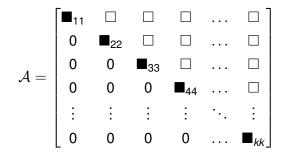


 $\mathcal{A}_{_{\textit{i}\textit{i}}} \stackrel{\text{def}}{=} \{ \ \textit{\textbf{A}}_{_{\textit{i}\textit{i}}} \ | \ \textit{\textbf{A}} \in \mathcal{A} \ \} = \mathbb{M}_{\textit{n}_{i}}(\mathbb{C}) \text{ or } \{\textit{\textbf{0}}_{_{1\times 1}}\}.$

Block-Upper- Forms

Definition

Let us call this a maximal block-upper- \triangle form of \mathcal{A} .

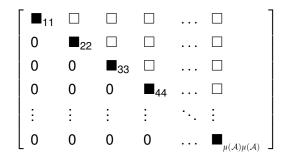


 $\mathcal{A}_{\textit{ii}} \stackrel{\text{def}}{=} \{ \text{ } \textbf{\textit{A}}_{\textit{ii}} \mid \textbf{\textit{A}} \in \mathcal{A} \} = \mathbb{M}_{\textit{n}_{i}}(\mathbb{C}) \text{ or } \{\textbf{0}_{1 \times 1}\}.$

 $\mu(\mathcal{A})$

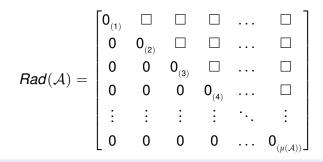
Notation

A maximal block-upper- \triangle form may not be unique, but the number *k* of the blocks on the block-diagonal is an intrinsic property of \mathcal{A} . Let us denote this *k* by $\mu(A)$.



Nil Radical

Since irreducible matrix algebras are simple, when \mathcal{A} is in a maximal block-upper- \triangle form, the nil radical $Rad(\mathcal{A})$ is exactly the subalgebra of all strictly block-upper- \triangle matrices in \mathcal{A} .

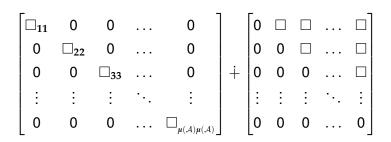


In particular $(Rad(A))^{\mu(A)} = \{\mathbf{0}_{n \times n}\}.$

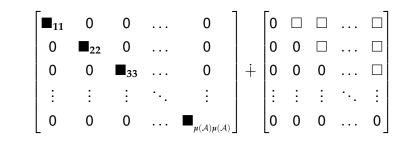
"Unhinged" Block-Upper- \triangle form

Terminology

A block-upper- \triangle subalgebra of $\mathbb{M}_n(\mathbb{C})$ is **unhinged** if the algebra is an internal direct sum of its block-diagonal subalgebra and its strictly block-upper- \triangle subalgebra.



If \mathcal{A} has an unhinged maximal block-upper- \triangle form, then it is clear that the block diagonal subalgebra is a Wedderburn factor of \mathcal{A} , and the strictly block-upper- \triangle subalgebra is the nil radical of \mathcal{A} .



"A Natural Question"

Question

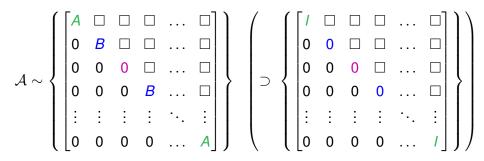
Given a matrix algebra \mathcal{A} in a maximal block-upper- \triangle form, is it always possible to unhinge \mathcal{A} without "messing up" the underlying spatial decomposition; i.e. via an application of a block-upper- \triangle similarity?

Barker-Eifler-Kezlan Theorem [1978]

(conjectured by H. Schneider)

Up to similarity, every subalgebra \mathcal{A} in $\mathbb{M}_n(\mathbb{C})$ has a maximal

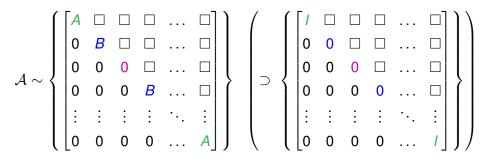
block-upper- \triangle form such that <u>the non-zero block-diagonal positions</u> are either "linked" or "independent".



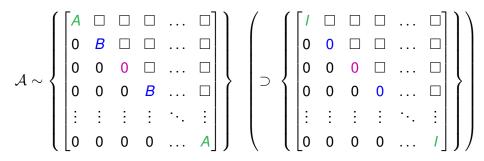
Let us call this a maximal grouped block-upper- \triangle form of \mathcal{A} .

J. F. Watters [1980]

Given a subalgebra \mathcal{A} of $\mathbb{M}_n(\mathbb{F})$ in a maximal block-upper- \triangle form, after some <u>block-diagonal similarity</u> one will arrive at an algebra in a maximal grouped block-upper- \triangle form.



Furthermore, <u>the block-diagonal</u> of a maximal grouped block-upper- \triangle form is fairly unique: if two such forms are similar, then (up to a block-diagonal similarity) the block-diagonal blocks are simply being permuted.



The answer to the "Natural Question" is affirmative when \mathcal{A} is semi-simple.

Theorem

Given a semi-simple subalgebra \mathcal{A} of $\mathbb{M}_n(\mathbb{F})$ in a maximal block-upper- \triangle form, there is a <u>block-upper- \triangle similarity</u> which implements the compression to the block-diagonal of \mathcal{A} ;

i.e. there is an invertible *block-upper*- \triangle *T* such that

 $T^{-1}AT = BlockDiag(A)$, for all $A \in A$

Of course, via Barker-Eifler-Kezlan-Watters' theorem, we can insure that the block-diagonal algebra BlockDiag(A) thus obtained has a particularly simple grouped form.

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At this point we can also use a block-permutation similarity to "shuffle" the block-diagonal algebra so that the "linked" blocks become adjacent.

We have arrived at the following conclusion:

If \mathcal{A} is a semi-simple subalgebra of $\mathbb{M}_n(\mathbb{C})$, then \mathcal{A} is **simulatneously similar** to an internal direct sum

$$\bigoplus_{i} \left(\mathbb{M}_{k_{i}}(\mathbb{C}) \otimes I_{p_{i}} \right)$$

In fact more is true:

If \mathcal{A} is a semi-simple subalgebra of $\mathbb{M}_n(\mathbb{F})$, then there exist irreducible division algebras \mathcal{D}_i of matrices over \mathbb{F} , such that \mathcal{A} is **simulatneously similar** to an internal direct sum

$$\bigoplus_{i} \left(\mathbb{M}_{k_{i}}(\mathcal{D}_{i}) \otimes I_{p_{i}} \right)$$

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Compare this to the classical theorem of Wedderburn which states that a semi-simple finite-dimensional \mathbb{F} -algebra \mathcal{A} can be written, up to an algebra isomorphism (and quite uniquely), as a direct sum of full matrix algebras over division \mathbb{F} -algebras.

Affirmative General Answer to the "Natural Question"

Theorem

Given a subalgebra \mathcal{A} of $\mathbb{M}_n(\mathbb{C})$ in a maximal block-upper- \triangle form, after some <u>block-upper- \triangle similarity</u> one will arrive at an algebra in an

unhinged maximal grouped block-upper- \triangle form.

Affirmative General Answer to the "Natural Question"

Theorem

Given a subalgebra \mathcal{A} of $\mathbb{M}_n(\mathbb{C})$ in a maximal block-upper- \triangle form, after some <u>block-upper- \triangle similarity</u> one will arrive at an algebra in an **unhinged maximal grouped block-upper-** \triangle form.

Corollary

If S is <u>any</u> Wedderburn factor of a subalgebra A of $\mathbb{M}_n(\mathbb{C})$, then after an appropriate change of basis, A sports an unhinged maximal grouped block-upper- Δ form, where the elements of S are exactly the block-diagonal matrices.

These results are also true for algebras in a general field setting (but there are some restrictions).

A Neat Application

Recall that $\mu(A)$ stands for the number of blocks on the block-diagonal of any maximal block-upper- \triangle form of a matrix algebra A, and that $\left(\text{Rad}(A)\right)^{\mu(A)} = \{0\}$.

Theorem

If \mathbb{F} is <u>algebraically closed</u>, the following are equivalent for a subalgebra \mathcal{A} of $\mathbb{M}_{p}(\mathbb{F})$:

- $(Rad(\mathcal{A}))^{\mu(\mathcal{A})-1} \neq \{0\};$
- **2** There is an element $R \in Rad(A)$ such that $R^{\mu(A)-1} \neq 0$.

In such a case, A is unicellular, i.e. the lattice of the invariant subspaces of A is totally ordered by inclusion.

It is certainly known to the algebraists that the use of $\mu(A)$ is essential in the result above. For example, consider the nilpotent algebra

$$\mathcal{A} = \left\{ \left. egin{array}{cccc} 0 & a & b & c \ 0 & 0 & 0 & -b \ 0 & 0 & 0 & a \ 0 & 0 & 0 & 0 \end{array}
ight| \, egin{array}{cccc} a, b, c \in \mathbb{C} \ a, b, c \in \mathbb{C} \end{array}
ight\}.$$

It is clear that $\mathcal{A}^3=\{0\},$ but $\mathcal{A}^2\neq\{0\}$ since

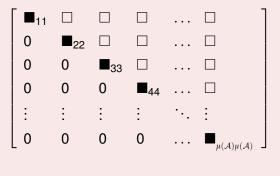
$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \circ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \neq 0.$$

Yet it is easy to check that $T^2 = 0$ for every $T \in A$.

Essential Lemma 1 (Back to (1, d)-Transitivity ...)

Lemma

If a minimal (1, *d*)-transitive subalgebra \mathcal{A} of $\mathcal{L}(\mathcal{V})$ has a maximal block-upper- \triangle form



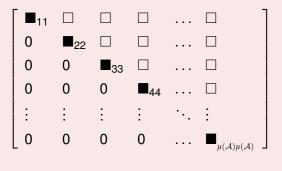
then

$$n_i \leq n-d+1$$
, for all i .

Essential Lemma 1 (a different formulation)

Lemma

If a subalgebra \mathcal{A} of $\mathcal{L}(\mathcal{V})$, minimal with respect to the property $\mathcal{A}\langle 1 \rangle \geq \langle p \rangle$, is expressed in a maximal block-upper- \triangle form



then

 $n_i \leq p$, for all *i*.

Essential Lemma 2

Lemma

If \mathcal{A} is a <u>minimal</u> (1, d)-transitive subalgebra of $\mathcal{L}(\mathcal{V})$, then the invariant subspaces of co-dimension d - 1 are exactly the minimal non-trivial invariant subspaces of \mathcal{A} .

Consequently, there is an x such that

$$\operatorname{co-dim}(\mathcal{A}x) = d - 1.$$

Essential Lemma 2 (alternate formulation)

Lemma

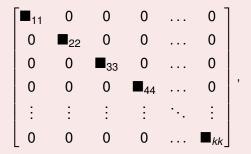
If a subalgebra \mathcal{A} of $\mathcal{L}(\mathcal{V})$ is minimal with respect to the property $\mathcal{A}\langle 1 \rangle \geq \langle p \rangle$, then the invariant subspaces of dimension p are exactly the minimal non-trivial invariant subspaces of \mathcal{A} .

Consequently, there is an x such that

 $dim(\mathcal{A}x) = p.$

Semi-Simple (1, *d*)-Transitive Matrix Algebras

If a subalgebra $\mathcal A$ of $\mathbb M_n(\mathbb C)$ has a maximal block-diagonal form

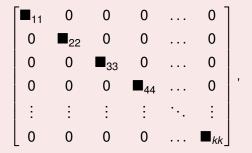


then

 \mathcal{A} is (1, d)-transitive if and only if $n_i > n - d$, for all *i*.

Semi-Simple (1, *d*)-Transitive Matrix Algebras

If a subalgebra $\mathcal A$ of $\mathbb M_n(\mathbb C)$ has a maximal block-diagonal form



then

$$\mathcal{A}\langle 1 \rangle \geq \langle p \rangle$$
 if and only if $n_i \geq p$, for all *i*.

Semi-Simple Minimal (1, d)-Transitive Algebras

Theorem

A subalgebra \mathcal{A} of $\mathbb{M}_n(\mathbb{C})$ in a maximal grouped block-diagonal form is <u>minimal</u> (1, *d*)-transitive \downarrow^a if and only if $d = n - \frac{n}{k} + 1$ and

$$\mathcal{A} = \left\{ \begin{array}{ccccc} A & 0 & 0 & 0 & \dots & 0 \\ 0 & A & 0 & 0 & \dots & 0 \\ 0 & 0 & A & 0 & \dots & 0 \\ 0 & 0 & 0 & A & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & A \end{array} \right| \quad A \in \mathbb{M}_{\frac{n}{k}}(\mathbb{C}) \\ \end{array} \right\} = \mathbb{M}_{\frac{n}{k}}(\mathbb{C}) \otimes I_{k}$$

^a $1 \leq d < n$

Semi-Simple Minimal (1, d)-Transitive Algebras

Theorem

A subalgebra \mathcal{A} of $\mathbb{M}_n(\mathbb{C})$ in a maximal grouped block-diagonal form is <u>minimal</u> with respect to the property $\mathcal{A}\langle 1 \rangle \geq \langle p \rangle$ if and only if $(1 \neq) p | n$ and

$$\mathcal{A} = \left\{ \begin{array}{cccccc} A & 0 & 0 & 0 & \dots & 0 \\ 0 & A & 0 & 0 & \dots & 0 \\ 0 & 0 & A & 0 & \dots & 0 \\ 0 & 0 & 0 & A & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & A \end{array} \right| \quad A \in \mathbb{M}_{p}(\mathbb{C}) \right\} = \mathbb{M}_{p}(\mathbb{C}) \otimes I_{n/p}$$

i.e.

A <u>minimal</u> (1, d)-transitive subalgebra \mathcal{A} of $\mathbb{M}_n(\mathbb{C}) \downarrow^a$ is semi-simple if and only if $d = n - \frac{n}{k} + 1$ and \mathcal{A} is simultaneously similar to $\mathbb{M}_{\frac{n}{k}}(\mathbb{C}) \otimes I_k$.

^a $1 \le d < n$

i.e.

A semi-simple subalgebra \mathcal{A} of $\mathbb{M}_n(\mathbb{C})$ is <u>minimal</u> with respect to the property $\mathcal{A}\langle 1 \rangle \geq \langle p \rangle$ if and only if $(1 \neq) p | n$ and \mathcal{A} is simultaneously similar to $\mathbb{M}_p(\mathbb{C}) \otimes I_{n/p}$

Canonical Minimal (1, *d*)-Transitive Types

Recall that we have seen one canonical type of a minimal (1, d)-transitive matrix algebra so far:

"The Usual Suspects" type (not semi-simple)

$$\mathcal{A} = \left\{ \begin{array}{c|c} A & B \\ 0 & 0 \end{array} \right| \ A \in \mathbb{M}_{_{n-(d-1)}}(\mathbb{C}); \ B \in \mathbb{M}_{_{(n-(d-1))\times(d-1)}}(\mathbb{C}) \end{array} \right\} \subset \mathbb{M}_{n}(\mathbb{C})$$

Of course these algebras are in a maximal block-upper- \triangle form, and have a non-trivial nil radical.

Canonical Minimal (1, *d*)-Transitive Types

"The Usual Suspects" type (not semi-simple)

$$\mathcal{A} = \left\{ \begin{array}{c|c} A & B \\ 0 & 0 \end{array} \right| \ A \in \mathbb{M}_{n-(d-1)}(\mathbb{C}); \ B \in \mathbb{M}_{(n-(d-1))\times(d-1)}(\mathbb{C}) \end{array} \right\} \subset \mathbb{M}_{n}(\mathbb{C})$$

Now we also have a canonical semi-simple type:

"The New Kids on the Block (-Diagonal)" type (semi-simple)

$$\begin{array}{ll} d = 1 & : & \mathcal{A} = \mathbb{M}_n(\mathbb{C}) & (\text{``smallest size''; Burnside's}) \\ d = \frac{n}{2} + 1 & : & \mathcal{A} = \mathbb{M}_{\frac{n}{2}}(\mathbb{C}) \otimes I_2 & (\text{``just over half the size''}) \\ d = \frac{2n}{3} + 1 & : & \mathcal{A} = \mathbb{M}_{\frac{n}{3}}(\mathbb{C}) \otimes I_3 & (\text{``just over two-thirds the size''}) \\ \text{etc.} \end{array}$$

"Para-Burnside's" For "Smaller" d's.

Theorem

If $d \leq \lfloor \frac{n}{2} \rfloor$ then the minimal (1, *d*)-transitive matrix algebras are exactly "the usual suspects".

Theorem

If $d = \frac{n}{2} + 1$, then then the minimal (1, d)-transitive matrix algebras are exactly "the usual suspects" and "the new kids on the block": $(\mathbb{M}_{\frac{n}{2}}(\mathbb{C}) \otimes I_2).$

"Para-Burnside's" For "Smaller" d's.

Theorem

If $p > \lfloor \frac{n}{2} \rfloor$ then the algebras minimal with respect to the property $\mathcal{A}\langle 1 \rangle \geq \langle p \rangle$ are exactly "the usual suspects".

Theorem

If $p = \frac{n}{2}$ then the algebras minimal with respect to the property $\mathcal{A}\langle 1 \rangle \geq \langle p \rangle$ are exactly "the usual suspects" and "the new kids on the block": $(\mathbb{M}_p(\mathbb{C}) \otimes I_2)$.

Still open:

What are the minimal (1, d)-transitive algebras for slightly bigger d's?

Basic Observations

The following are equivalent for a subalgebra $\mathcal A$ of $\mathcal L(\mathcal V)$ and $d \leq \textit{dim}(\mathcal V) \text{:}$

- \mathcal{A} is (2, *d*)-transitive;
- ② For every linearly independent pair x, y: co-dim(Ax + Ay) < d;
- Every non-trivial invariant subspace of A is either one-dimensional or has co-dimension less than d.

Basic Observations

The following are equivalent for a subalgebra $\mathcal A$ of $\mathcal L(\mathcal V)$ and $d \leq \textit{dim}(\mathcal V) \text{:}$

- \mathcal{A} is (2, *d*)-transitive;
- ② For every linearly independent pair x, y: co-dim(Ax + Ay) < d;
- Every non-trivial invariant subspace of A is either one-dimensional or has co-dimension less than d.

Of course, every (1, d)-transitive algebra is automatically (2, d)-transitive.

If $d = dim(\mathcal{V})$ then \mathcal{A} is (2, *d*)-transitive if and only if the common kernel of \mathcal{A} is at most 1-dimensional. So, again, we shall only consider the case $d \le dim(\mathcal{V}) - 1$ henceforth.

An Essential Lemma

Unfortunately the direct analogues of the Essential Lemmas for (1, d)-transitivity do not hold true for (2, d)-transitivity. Fortunately there is still something we can say in this case.

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Essential Lemma

If d < dim(V) - 1 then every (2, *d*)-transitive subalgebra of $\mathcal{L}(V)$ has at most one 1-dimensional invariant subspace.

If $d \leq \left\lceil \frac{\dim(\mathcal{V})}{2} \right\rceil$, then every minimal (2, *d*)-transitive subalgebra \mathcal{A} of $\mathcal{L}(\mathcal{V})$ has exactly one 1-dimensional invariant subspace.

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i.e. if $p > \left\lfloor \frac{\dim(\mathcal{V})}{2} \right\rfloor$ then minimal $\mathcal{A}\langle 2 \rangle \ge \langle p \rangle$ algebras have exactly one 1-dimensional invariant subspace.

"Para-Burnside's" for Smaller d's and Larger p's

Theorem

If $d \leq \lfloor \frac{\dim(\mathcal{V})}{2} \rfloor$, then the minimal (2, *d*)-transitive subalgebras of $\mathcal{L}(\mathcal{V})$ are exactly "the usual suspects".

"Para-Burnside's" for Smaller d's and Larger p's

Theorem

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In other words, for $p > \left\lceil \frac{\dim(\mathcal{V})}{2} \right\rceil$, the minimal $\mathcal{A}\langle 2 \rangle \geq \langle p \rangle$ subalgebras of $\mathcal{L}(\mathcal{V})$ are exactly the usual suspects.

Semi-Simple (2, *d*)-Transitive Algebras

Lemma (LL)

The following are equivalent for a semi-simple subalgebra \mathcal{A} of $\mathbb{M}_n(\mathbb{C})$:

- \mathcal{A} is (2, d)-transitive, but not (1, d)-transitive;
- $\begin{array}{|c|c|c|c|} \hline \textbf{O} & \underline{\text{Case "}d < n-1":} \ \mathcal{A} \text{ is simultaneously similar to either } \mathcal{B} \oplus \{\textbf{0}_{1\times 1}\} \\ & \text{or to } \mathcal{B} \oplus \mathbb{C}l_1, \text{ where } \mathcal{B} \text{ is a } (1, d-1) \text{-transitive semi-simple} \\ & \text{subalgebra of } \mathbb{M}_{n-1}(\mathbb{C}); \quad (\text{i.e. } \mathcal{B}\langle 1 \rangle \geq \langle n-d+1 \rangle) \ . \end{array}$
 - ② <u>Case "*d* = *n* − 1":</u> *A* is simultaneously similar to either $\mathcal{B} \oplus \{0_{1\times 1}\}$ or to $\mathcal{B} \oplus \mathbb{C}I_{n_2} \oplus \ldots \oplus \mathbb{C}I_{n_k}$ ($k \ge 2$), where \mathcal{B} is a (semi-simple) algebra such that $\mathcal{B}\langle 1 \rangle \ge \langle 2 \rangle$, and it may be absent in the second case.

Semi-Simple Minimal (2, d)-Transitive Algebras

Theorem (LL)

1. ℂ*I*

Up to similarity, the full list of the minimal (2, d)-transitive subalgebras \downarrow^a that are semi-simple, is this:

minimal with respect to:

$$\mathcal{A}\langle \mathbf{2}
angle \geq \langle \mathbf{2}
angle;$$

4. $\mathbb{M}_m(\mathbb{C}) \otimes I_k$ 2 < m & 1 < k $\mathcal{A}\langle 2 \rangle \ge \langle m \rangle;$ 5. $(\mathbb{M}_m(\mathbb{C}) \otimes I_k) \oplus \{0\}$ 2 $\leq m$ $\mathcal{A}\langle 2 \rangle \ge \langle m \rangle;$

^{*a*} $1 \le d < n =$ the dimension of the underlying space, which is at least 3.

Still Open

What are the minimal (2, d)-transitive algebras for slightly bigger *d*'s (slightly smaller *p*'s)?

For the rest of the talk, we shall include the case $dim\mathcal{V}=2$ in the consideration.

We can completely classify the subalgebras $\mathcal A$ of $\mathcal L(\mathcal V)$ such that:

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2 \mathcal{A} is minimal with respect to the property $\mathcal{A}\langle 2 \rangle = \langle m \rangle$;

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We can completely classify the subalgebras $\mathcal A$ of $\mathcal L(\mathcal V)$ such that:

- 2 \mathcal{A} is minimal with respect to the property $\mathcal{A}\langle 2 \rangle = \langle m \rangle$;
- 3 \mathcal{A} is maximal with respect to the property $\mathcal{A}\langle k \rangle = \langle m \rangle$;

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We can completely classify the subalgebras $\mathcal A$ of $\mathcal L(\mathcal V)$ such that:

- 2 \mathcal{A} is minimal with respect to the property $\mathcal{A}\langle 2 \rangle = \langle m \rangle$;
- 3 \mathcal{A} is maximal with respect to the property $\mathcal{A}\langle k \rangle = \langle m \rangle$;
- \mathcal{A} has the property $\mathcal{A}\langle k \rangle \leq \langle m \rangle$, where $k \geq m$.

"The Exactly Usual Suspects"

Theorem

If $\mathcal W,\,\mathcal Z$ are subspaces of $\mathcal V$ such that

$$0 \dim(\mathcal{Z}) = k - 1;$$

2 dim $(\mathcal{W}) = m$;

then the algebra \mathcal{A} of all those linear transformations on \mathcal{V} which map \mathcal{V} into \mathcal{W} , and vanish on \mathcal{Z} , satisfies the condition

$$\mathcal{A}\langle \mathbf{k}\rangle = \langle \mathbf{m}\rangle.$$

Terminology

The algebras of the type described in this theorem shall be denoted by $\mathcal{A}_{e}(\mathcal{Z}; \mathcal{W})$ and referred to as "the exactly usual suspects".

The Minimal Exactly Usual Suspects

Theorem

If $\mathcal W,\,\mathcal Z$ are subspaces of $\mathcal V$ such that

$$\bigcirc \dim(\mathcal{Z}) = k - 1;$$

$$olim(\mathcal{W}) = m;$$

then $\mathcal{A}_e(\mathcal{Z}; \mathcal{W})$ is <u>minimal</u> such that $\mathcal{A}\langle k \rangle = \langle m \rangle$ if and only if at least one of the following holds:

• dim
$$(W) = 1$$
; (i.e. $m = 1$)

$$c_0 - d_{(m}(z) = |$$

 $(j_2, k = d_{(m)}/v)$

• \mathcal{W} is not a subspace of \mathcal{Z} ; (i.e. $\mathcal{A}_{e}(\mathcal{Z}; \mathcal{W})$ is not nilpotent of order 2).

Obviously the possibility of non-minimality only comes into play when k > m > 1.

$$\mathcal{A}\langle \mathbf{1}
angle = \langle m
angle$$

Theorem

Subalgebras \mathcal{A} of $\mathcal{L}(\mathcal{V})$ that satisfy

$$\mathcal{A}\langle 1 \rangle = \langle m \rangle$$

are the exactly usual suspects,

except in the case m = 1, where the algebra $\mathbb{C}I_{\nu}$ should be added to the list.

Minimal $\mathcal{A}\langle 2 \rangle = \langle m \rangle$, for $m \neq 2$

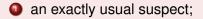
Theorem

For $m \neq 2$, the minimal $A\langle 2 \rangle = \langle m \rangle$ subalgebras of $\mathcal{L}(\mathcal{V})$ are (up to similarity, of course) exactly the exactly usual suspects.

Minimal $\mathcal{A}\langle 2 \rangle = \langle 2 \rangle$

Theorem

A subalgebra \mathcal{A} of $\mathcal{L}(\mathcal{V})$ is <u>minimal</u> with respect to the condition $\mathcal{A}\langle 2 \rangle = \langle 2 \rangle$ if and only if it is (up to similarity, of course) one of the following:



$$\mathbf{\mathfrak{3}} \ \mathcal{A} = \left\{ \begin{array}{cc} \left[\alpha I_{\mathcal{W}} & T \\ \mathbf{0} & \mathbf{0} \end{array} \right] \middle| \ \alpha \in \mathbb{C}, \ T \in \mathcal{M} \right\},$$

where dim $\mathcal{V} > 2 = \dim \mathcal{W}$.

Here \mathcal{A} is being expressed with respect to a decomposition $\mathcal{V} = \mathcal{W} \dotplus \mathcal{Z}$, and \mathcal{M} is a minimal transitive subspace of $\mathcal{L}(\mathcal{Z}, \mathcal{W})$.

Maximal $\mathcal{A}\langle k \rangle = \langle m \rangle$

Theorem

Subalgebras \mathcal{A} of $\mathcal{L}(\mathcal{V})$ that are <u>maximal</u> with respect to the property

 $\mathcal{A}\langle \mathbf{k}
angle = \langle \mathbf{m}
angle$

are exactly the algebras $\mathcal{L}(\mathcal{V}, \mathcal{W})$, where \mathcal{W} is an *m*-dimensional subspace of \mathcal{V} ,

except in the case m = k, where the algebra $\mathbb{C}I_{v}$ should be added to the list.

$\mathcal{A}\langle k \rangle \leq \langle m \rangle$, where $k \geq m$

Theorem

Given $k \ge m$, the subalgebras \mathcal{A} of $\mathcal{L}(\mathcal{V})$ that satisfy

 $\mathcal{A}\langle \mathbf{k}\rangle\leq\langle m
angle$,

are exactly the subalgebras of the algebras $\mathcal{L}(\mathcal{V}, \mathcal{W})$ where \mathcal{W} is an *m*-dimensional subspace of \mathcal{V} ,

except in the case m = k where the algebras $\mathbb{C}E$ should be added to the list, for all non-zero idempotents *E*.

Thanks for your attention!

Paratransitivity for Algebras of Linear Transformations

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June 9, 2014

LAW 2014; Ljubljana, Slovenia

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