Sensitivity Analysis for Perfect State Transfer in Quantum Walks

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A model for quantum communication

We consider a model for quantum communication originally described by Bose (2003). Bose describes a type of *quantum wire* for transferring information, and proposes it as a simple quantum bus within a quantum computer.

Setup: We have an undirected graph on n vertices – vertices represent spins, and edges connect spins that interact. Select two vertices, a sender (vertex s) and a receiver (vertex r). A quantum state is input at vertex s. By engineering physical couplings between the spins, the spins can be made to interact with each other. After a time t has elapsed, the unknown state is communicated to vertex r, where it's read out, and is, we hope, close to the input state. The graph–theoretic distance from s to r (i.e. the number of edges on a shortest path from s to r) is a proxy for the physical distance travelled along our quantum wire.

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Fidelity

The quality of the communication is measured by the *fidelity*, which a number between 0 and 1. The fidelity at time $t, f(t)$, measures the similarity between the sending state and the receiving state: $f(t) = 1$ corresponds to exact communication of the state, $f(t)$ near to zero corresponds to poor communication.

Under certain reasonable physical hypotheses (uniform coupling on an unmodulated chain with XY Hamiltonian), the dynamics of the communication are described by a continuous-time quantum walk on the unweighted graph modeling the network.

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Fidelity for mathematicians

Suppose that we have a graph G on vertices labelled $1, \ldots, n$. The *adjacency matrix* for G, say A, is the $n \times n$ $(0, 1)$ matrix with $a_{p,q} = 1$ if $p \neq q$ and vertices p and q are adjacent, and $a_{p,q} = 0$ otherwise.

For a graph with adjacency matrix A, sender vertex s and receiver vertex r, the fidelity at time t is given by $f(t) = |exp(itA)_{s,r}|$. It is straightforward to show that for any $t, \sum_{r=1}^{n} |exp(itA)_{s,r}|^2 = 1$.

Hence, the quantity $p_{s,r}(t) \equiv f(t)^2$ can be thought of as the probability that a quantum walk on the graph starting vertex s arrives at vertex r at time t.

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A variant

The Laplacian matrix for a graph G is given by $L = D - A$, where A is the adjacency matrix of G and D is the diagonal matrix of row sums of A.

Under a modification of our earlier hypothesis – XX dynamics instead of XY dynamics – then the fidelity at time t is given by $\hat{f}(t) = |exp(itL)_{s,r}|$. As before, $\hat{p}_{s,r}(t) \equiv \hat{f}(t)^2$ is the probability that a (different) quantum walk on the graph starting vertex s arrives at vertex r at time t.

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Perfect state transfer

Suppose that at some time t_0 we have $p_{s,r}(t_0) = 1$ (or alternatively, $\hat{p}_{s,r}(t_0) = 1$). When this occurs we say that there is *perfect state transfer* from vertex s to vertex r at time t_0.

Example 1: Suppose that $d \in \mathbb{N}$, and consider the d–cube: vertices are the $(0,1)$–vectors in $\mathbb{R}^d$, with two vertices adjacent precisely when the corresponding vectors differ in exactly one position. For XY (i.e. adjacency matrix) dynamics, it is straightforward to show that for the d–cube, at time $t_0 = \frac{\pi}{2}$, there is perfect state transfer between any two vertices at distance d.

Example 2: The threshold graphs are well–studied and highly structured and are constructed inductively via a sequence of unions or joins with isolated vertices. There is a subfamily of threshold graphs for which, using XX (i.e. Laplacian matrix) dynamics, there is PST at time $t_0 = \frac{\pi}{2}$ between a particular pair of vertices.

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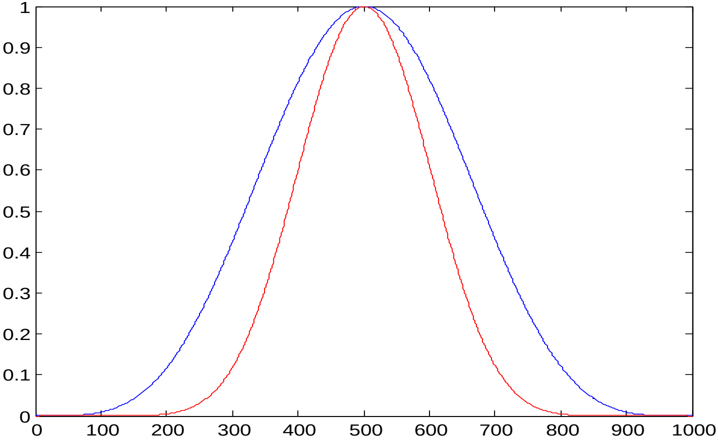
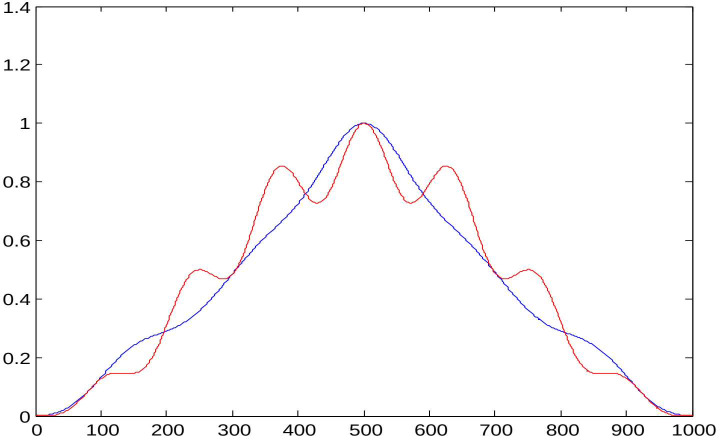
$p_{s,r}(t), t \in [0, \pi]$ for the 2–cube and the 5–cube



$\hat{p}_{s,r}(t), t \in [0, \pi]$ for two threshold graphs on 16 vertices



Two potential implementation problems

There are by now plenty of papers that construct graphs with PST. For example, Bose's original paper has more than 800 citations, and many of those citing papers identify new families of graphs with PST.

With so many PST graphs to choose from, perhaps other considerations should be brought to bear in selecting among the graphs with PST.

Question 1: What if my watch doesn't work? How sensitive is $p_{s,r}(t)$ to the value of t in a neighbourhood of t_0?

Question 2: What if one of my magnets doesn't work? How sensitive is $p_{s,r}(t_0)$ to small changes in the weights of the edges?

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A handy fact

Let M be a symmetric matrix of order n, (M will eventually be taken to be the adjacency matrix or the Laplacian matrix, as needed) and suppose that $M = V\Lambda V^T$, where Λ is a diagonal matrix of eigenvalues, and V is a corresponding orthogonal matrix whose columns are eigenvectors.
Set $U(t) = exp(itM)$ and note that $|U(t)_{s,r}|^2 = |e_s^T U(t) e_r|^2 = |e_s^T Vexp(it\Lambda)V^T e_r|^2 \leq ||e_s^T Vexp(it\Lambda)||_2 ||V^T e_r||_2 \leq 1$. (Cauchy–Schwarz.)

In particular, if $|U(t_0)_{s,r}|^2 = 1$ then necessarily there is a complex number γ such that $e_r^T V = \gamma e_s^T Vexp(it\Lambda)$; necessarily $|\gamma| = 1$.

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Derivatives with respect to readout time

For each $k \in \mathbb{N}$, let $w(k)$ denote the (s,s) entry in M^k.

Using the handy fact, it's straightforward to see that $\frac{d^k U(t)_{s,r}}{dt^k}|_{t_0} = (i)^k e_s^T V \Lambda^k exp(it_0 \Lambda) V^T e_r = \overline{\gamma}(i)^k e_s^T V \Lambda^k V^T e_s = \overline{\gamma}(i)^k w(k)$.

We then get the following result for the adjacency matrix (XY dynamics).

Theorem

Let G be a graph on n vertices, and suppose that $p_{s,r}(t_0) = 1$. Fix $k \in \mathbb{N}$. If k is odd, then $\frac{d^k p_{s,r}(t)}{dt^k}|_{t_0} = 0$. If k is even, then $\frac{d^k p_{s,r}(t)}{dt^k}|_{t_0} = (-1)^{(k \mod 4)/2} \sum_{j=0}^{k} (-1)^j \binom{k}{j} w(j) w(k-j)$.

For the Laplacian version (XX dynamics), add hats.

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For the Laplacian version (XX dynamics), add hats.

Derivatives with respect to readout time

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In particular, we learn that if we have PST from s to t at time t_0, then $\frac{dp_{s,r}(t)}{dt}|_{t_0} = 0$, (no points), and that $\frac{d^2 p_{s,r}(t)}{dt^2}|_{t_0} = -2w(2)$.

This last helps to explain the earlier figures with the two cubes. For the 2–cube, $-2w(2) = -4$ and for the 5–cube, $-2w(2) = -10$.

We also get insight into the figure for the two threshold graphs. For one of those we have $-2\hat{w}(2) = -60$ and for the other we have $-2\hat{w}(2) = -420$.

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Taylor's theorem with remainder

Suppose that we have PST from s to r at time t_0, and suppose that the adjacency matrix has spectral radius ρ. Some simple estimates yield the fact that for any t, $|\frac{d^j p_{s,r}(t)}{dt^j}| \le 2^{j+1}\rho^j$. Then for any $k \in \mathbb{N}$, we have

$$p_{s,r}(t_0+h) =$$

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where $|R_{2k+2}| \le 2(2\rho)^{2k+2}$.

In view of the above, we might take a special interest in graphs with PST for which ρ is not too large. For the Laplacian case, add hats as needed.

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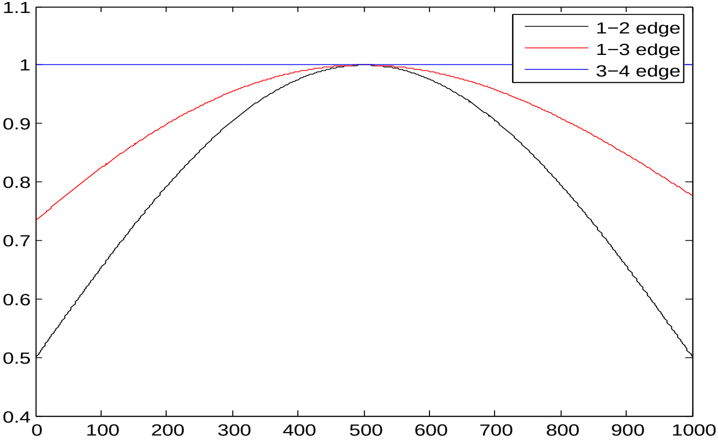
$\hat{p}_{s,r}(\frac{\pi}{2})$ for a particular threshold graph on 8 vertices, with edge weights perturbed by $x \in [-0.5, 0.5]$



Some analysis

Let G be a graph with adjacency matrix A, and select a pair of indices j, k. Set $E = e_j e_k^T + e_k e_j^T$, and let $A(x) = A + xE$, which is the adjacency matrix of the weighted graph formed from G by weighting the edge between vertices k and j with the value $a_{k,j} + x$. Let L be the Laplacian matrix of G and $\hat{E} = (e_j - e_k)(e_j - e_k)^T$, and set $L(x) = L + x\hat{E}$. Then $L(x)$ is the Laplacian matrix of the weighted graph formed from G by weighting the edge between vertices k and j with the value $-l_{k,j} + x$.

Fact: There is neighbourhood of 0, an orthogonal matrix $V(x)$, and a diagonal matrix $\Lambda(x)$, such that for each x in that neighbourhood, $V(x)$ and $\Lambda(x)$ are analytic in x and $A(x) = V(x)\Lambda(x)V(x)^T$. Similarly, there is an orthogonal matrix $\hat{V}(x)$ and a diagonal matrix $\hat{\Lambda}(x)$, both differentiable in x on a neighbourhood of 0 such that $L(x) = \hat{V}(x)\hat{\Lambda}(x)\hat{V}(x)^T$.

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Theorem

Suppose that $\overline{\gamma} = \alpha + i\beta$. Then

$$\begin{aligned} q''(0) = & -2t_0^2\{e_s^T V(0)(\Lambda'(0))^2 V(0)^T e_s - (e_s^T V(0)\Lambda'(0)V(0)^T e_s)^2\} \\ & -2\{e_s^T V'(0)(V'(0))^T e_s + e_r^T V'(0)(V'(0))^T e_r \\ & -2e_s^T V'(0)(\alpha\cos(t_0\Lambda) + \beta\sin(t_0\Lambda))(V'(0))^T e_r\}. \end{aligned}$$

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$$\begin{aligned} q''(0) = & -2t_0^2\{e_s^T V(0)(\Lambda'(0))^2 V(0)^T e_s - (e_s^T V(0)\Lambda'(0)V(0)^T e_s)^2\} \\ & -2\{e_s^T V'(0)(V'(0))^T e_s + e_r^T V'(0)(V'(0))^T e_r \\ & -2e_s^T V'(0)(\alpha\cos(t_0\Lambda) + \beta\sin(t_0\Lambda))(V'(0))^T e_r\}. \end{aligned}$$

Remarks on $q''(0)$

1. It is straightforward to show that both of the terms in braces in the preceding theorem are nonnegative.
2. To cover the Laplacian (XX) case, add hats as needed.
3. There is an algorithm for finding $V(0), V'(0)$ and $\Lambda'(0)$. Construct $V(0)$ by considering eigenvectors that are in the null space of E, then taking certain linear combinations of those that aren't. The entries in $\Lambda'(0)$ can be found from $V(0)$. The columns of $V'(0)$ involve expressions of the form $(\lambda I - A)^{\dagger} EV(0)e_m$, where λ is an eigenvalue with corresponding eigenvector $V(0)e_m$.
4. There is an analogous algorithm for finding $\hat{V}(0), \hat{V}'(0)$ and $\hat{\Lambda}'(0)$. It's actually more straightforward than the one in 4 above.
5. It's possible to produce an upper bound on $|q''(0)|$ in terms of the quantity $\max \frac{1}{|\lambda_a - \lambda_b|}$, where the maximum is taken over distinct eigenvalues λ_a, λ_b of A (or L). This may help to guide the choice of graphs exhibiting PST.

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Example revisited

We saw a figure for a threshold graph on 8 vertices, and three curves corresponding to perturbation of the weights of the edge $1-2$, the edge $1-3$, and the edge $3-4$.

Using the preceding theorem, we can compute $q''(0)$ with respect to each edge, and obtain the following:
for $1-2$, $q''(0) = -\frac{\pi^2}{2}$;
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Final thoughts

Since $V(x)$ and $\Lambda(x)$ are analytic in a neighbourood of 0, higher order derivatives of q are available. However, the expressions become uglier with the number of derivatives taken.

The results on readout time suggest that sparse graphs with PST will offer some forgiveness for small errors in readout time.

If we have a graph in hand exhibiting PST, the results on the second derivatives of $p_{s,r}$ with respect to the various edge weights let us know which edges exert more influence on the fidelity. This might help to inform the implementation of the corresponding quantum walk.

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