Riesz distributions on symmetric matrices and the Olkin-Rubin Characterization

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Introduction
Riesz distribution
Riesz inverse Gaussian distributions
Riesz-Dirichlet distribution
Beta-Riesz distribution
Olkin-Rubin theorem

V the space of real symmetric $r \times r$ matrices
Ω the cone of positive definite elements of V
e the identity matrix
$\Delta(x)$=determinant of x

Notion of "quotient" on symmetric matrices

$y \in \Omega$, $y = y^{\frac{1}{2}} y^{\frac{1}{2}}$ and define the ratio x by y as

$$P(y^{-\frac{1}{2}})(x) = y^{-\frac{1}{2}} x y^{-\frac{1}{2}}.$$

Cholesky decomposition $y = tt'$, where t is a lower triangular matrix t' is its transpose. We set

$$y(x) = txt'$$

The "quotient" of x by y is then defined as

$$y^{-1}(x) = t^{-1} x t'^{-1}$$

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Generalized power and spherical Fourier transform
Let $x = (x_{ij})_{1 \leq i,j \leq r}$ be in Ω.
For $1 \leq k \leq r$, we denote $\Delta_k(x) = \det((x_{ij})_{1 \leq i,\ j \leq k})$
The generalized power of X is then defined for $s = (s_1,...,s_r)$, by

$$\Delta_s(x) = \Delta_1(x)^{s_1 - s_2} \Delta_2(x)^{s_2 - s_3} \cdots \Delta_r(x)^{s_r}$$

In particular, when $s = e_i = (0,...,1,...,0)$, and with $\Delta_0(x) = 1$,

$$\Delta_{e_i}(x) = \frac{\Delta_i(x)}{\Delta_{i-1}(x)}.$$

For x and y in Ω_r, we have

$$\Delta_s(y(x)) = \Delta_s(y)\Delta_s(x)$$

and

$$\Delta_s(y^{-1}(x)) = \Delta_s(x)\Delta_{-s}(y).$$

Spherical Fourier transform for a K-invariant distribution on Ω:

$$s \mapsto \mathbb{E}(\Delta_s(X))$$

It characterizes the distribution.....
It plays the role of the Mellin transform for the class of K-invariant distributions on Ω.
For X in Ω, we associate the vector

$$M(X) = (\Delta_{e_1}(X),...,\Delta_{e_r}(X)).$$

We have

$$\Delta_s(X) = (M(X))^s = \Delta_{e_1}(X)^{s_1}...\Delta_{e_r}(X)^{s_r}.$$

$$\mathbb{E}(\Delta_s(X)) = \mathbb{E}((M(X))^s)$$

In the real case ($r = 1$), $M(X) = X$.

For any matrix $x \in V$ and a given $k \in \{1, \cdots, r-1\}$, define the partitioning into blocks

$$x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$$

with x_{11} a $k \times k$ block, $x_{12} = x_{21}^*$ a $k \times (r-k)$ block and x_{22} a $(r-k) \times (r-k)$ block.

$$x_{1.2} = x_{11} - x_{12} x_{22}^{-1} x_{21} \text{ and } x_{2.1} = x_{22} - x_{21} x_{11}^{-1} x_{12}.$$

(Massam and Neher 1997) For $x \in \Omega$, we have

$$\det(x) = \det(x_{11}) \det(x_{2.1})$$

For $x \in \Omega$, we have

$$\Delta_s(x) = \Delta_{w_1}(x_{11}) \Delta_{w_2}(x_{2.1}),$$

where $w_1 = (s_1, \cdots, s_k)$ and $w_2 = (s_{k+1}, \cdots, s_r)$.

Riesz distribution

Consider the Gindikin set $\Lambda = \{\frac{1}{2}, 1, \cdots, \frac{(r-1)}{2}\} \cup (\frac{(r-1)}{2}, +\infty)$. For $p \in \Lambda$ it is well known (Gindikin, 1964) that there exists a positive measure μ_p on Ω such that the Laplace transform of μ_p exists for $-\theta \in \Omega$ and is equal to

$$L_{\mu_p}(\theta) = \int_{\Omega_r} e^{\langle \theta, x \rangle} \mu_p(dx) = (\det(-\theta))^{-p}.$$

We introduce the set Ξ of elements $s = (s_1, \cdots, s_r)$ of $\mathbb{R}^r$ defined as follows: For a real number $u \geq 0$, we put

$$\begin{array}{rcl} \varepsilon(u) & = & 0 \quad \text{if} \quad u = 0, \\ \varepsilon(u) & = & 1 \quad \text{if} \quad u > 0. \end{array}$$

For $u_1, u_2, \cdots, u_r \geq 0$, we define

$$\begin{array}{rcl} s_1 & = & u_1, \\ s_k & = & u_k + \frac{d}{2}(\varepsilon(u_1) + \cdots + \varepsilon(u_{k-1})), \quad 2 \leq k \leq r. \end{array}$$

Theorem 1. *There exists a positive measure R_s on V such that the Laplace transform is defined on $-\Omega_r$ and is equal to*

$$L_{R_s}(\theta) = \Delta_s(-\theta^{-1})$$

if and only if s is in Ξ.

Proposition 2. *Let $s \in \Xi$. The Riesz measure R_s is in $\mathcal{M}(V)$ if and only if $s_1 \neq 0$.*

Let $s \in \Xi$ such that $s_1 \neq 0$. $F(R_s)$ is called a Riesz N.E.F.

The Wishart on Ω, shape parameter $p > \frac{r-1}{2}$, scale parameter σ,

$$W_r(p,\sigma)(dx) = \frac{(\det \sigma)^p}{\Gamma_{\Omega_r}(p)} e^{-\langle \sigma, x\rangle} (\det x)^{p - \frac{r+1}{2}} \mathbf{1}_{\Omega_r}(x) dx,$$

where $\Gamma_\Omega(.)$ is the multivariate gamma function

$$\Gamma_\Omega(p) = (2\pi)^{\frac{r(r-1)}{2}} \prod_{k=1}^{r} \Gamma(p - \frac{k-1}{2}).$$

The absolutely continuous matrix Riesz distribution

$$R(s,\sigma)(dx) = \frac{1}{\Gamma_\Omega(s)\Delta_s(\sigma^{-1})} e^{-\langle \sigma, x\rangle} \Delta_{s-\frac{n}{r}}(x) \mathbf{1}_\Omega(x) dx,$$

where σ is in Ω, $s = (s_1, \cdots, s_r)$ is in $\mathbb{R}^r$ such that $s_i > (i-1)\frac{1}{2}$

$$\Gamma_\Omega(s) = (2\pi)^{\frac{n-r}{2}} \prod_{j=1}^{r} \Gamma(s_j - (j-1)\frac{1}{2}).$$

The Variance function

$P_i^*(x) = x_{22}$ of size i, for $1 \leq i \leq r$.

Theorem 3. *Let $F(R_s)$ be the Riesz NEF generated by the measure R_s. Then*

- $M_{F(R_s)} = \Omega$.
- $\forall m \in \Omega$,

$$V_{F(R_s)}(m) = \sum_{i=1}^{r} (s_{r-i+1} - s_{r-i}) P[\frac{1}{s_{r-i+1}} (P_i^*(m^{-1}))^{-1}$$

$$+ \sum_{k=1}^{i-1} (\frac{1}{s_{r-k+1}} - \frac{1}{s_{r-k}}) (P_k^*(m^{-1}))^{-1}].$$

Some properties.

Theorem 4. *Let X be a Riesz random variable on Ω_r with parameters (s,σ). Partition X and σ in blocks according to the dimension k and $r-k$ as*

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \qquad \sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}.$$

Then

- $X_{11} \sim R_k(w_1,\sigma_{1.2})$, *where* $w_1 = (s_1,\cdots,s_k)$.
- $X_{2.1} \sim R_{r-k}(w_2 - \frac{k}{2},\sigma_{22})$, *and is independent of* X_{12} *and* X_{11}, *where* $w_2 = (s_{k+1},\cdots,s_r)$.
- $X_{12}|X_{11} \sim N_{k\times(r-k)}\left(-\{\sigma_{22}^{-1}\sigma_{12}x_{11}\},(4L(\sigma_{22})L(x_{11}^{-1}))^{-1}\right)$.

Riesz inverse Gaussian distributions

The Riesz inverse Gaussian distribution on Ω is defined by

$$\frac{1}{\mathcal{K}(s,a,b)} \exp\{-(\langle a,x\rangle + \langle b,x^{-1}\rangle)\}\Delta_{s-\frac{r+1}{2}}(x)\mathbf{1}_{\Omega_r}(x)dx,$$

where $\mathcal{K}(s,a,b)$ is the generalized Bessel function and (s,a,b) satisfy

$$\begin{cases} b \in \overline{\Omega_r},\ a \in \Omega_r & \text{if} \quad s_i > \frac{(i-1)}{2},\ \forall 1 \leq i \leq r, \\ b \in \Omega_r,\ a \in \Omega_r & \text{if} \quad -\frac{(r-i)}{2} \leq s_i \leq \frac{(i-1)}{2},\ \forall 1 \leq i \leq r, \\ b \in \Omega_r,\ a \in \overline{\Omega_r} & \text{if} \quad s_i < -\frac{(r-i)}{2},\ \forall 1 \leq i \leq r. \end{cases}$$

Theorem 5. *Let X be a random variable in Ω with distribution $R_r(s,\sigma)$. Then the conditional distribution of X_{11} given X_{12} has a Riesz inverse Gaussian distribution with parameters $w_1 - \frac{r-k}{2}$, $2\sigma_{11}$ and $2P(x_{12})\sigma_{22}$, where $w_1 = (s_1, \cdots, s_k)$.*

$P(a)(x) = axa$

Riesz Dirichlet distribution

Theorem 6. *Let* $Y_1, \cdots, Y_q$ *be* q *independent Riesz random variables with the same* σ, $Y_j \sim R(s^j, \sigma)$, *where* $s^j = (s_1^j, \cdots, s_r^j)$ $\forall 1 \leq j \leq q$. *If we set* $S = Y_1 + \cdots + Y_q$ *and* $X_j = S^{-1}(Y_j)$, *then*

i) S *is a Riesz random variable* $S \sim R(\sum_{j=1}^{q} s^j, \sigma)$ *and is independent of* $(X_1, \cdots, X_{q-1})$.

ii) The density of the joint distribution of $(X_1, \cdots, X_{q-1})$

$$\frac{\Gamma_\Omega(\sum_{j=1}^{q} s^j)}{\prod_{j=1}^{q} \Gamma_\Omega(s^j)} \prod_{j=1}^{q-1} \Delta_{s^j - \frac{n}{r}}(x_j) \Delta_{s^q - \frac{n}{r}}(e - (x_1 + \cdots + x_{q-1})$$

where $x_j \in \Omega$, $1 \leq j \leq q-1$ *and* $e - \sum_{j=1}^{q-1} x_j \in \Omega$.

Definition 7. *The distribution of* $(X_1, \cdots, X_q)$ *is called the Riesz-Dirichlet distribution on* V *with parameters* $(s^1, \cdots, s^q)$ *denoted by* $D_{(s^1,\cdots,s^q)}$.

If $s_k^j = p_j$; $1 \leq k \leq r$, then $D_{(s^1,\cdots,s^q)}$ is Wishart-Dirichlet $D_{(p_1,\cdots,p_q)}$.

Theorem 8. *Let* $X = (X_1, \cdots, X_q)$ *be a Riesz-Dirichlet random variable with distribution* $D_{(s^1, \cdots, s^q)}$. *Then for all* $1 \le k \le r$, *the random variable* $(P_k(X_1), \cdots, P_k(X_q))$ *has a Dirichlet distribution* $D_{(\underline{s}^1, \cdots, \underline{s}^q)}$, *where* $\underline{s}^i = (s_1^i, \cdots, s_k^i)$, $\forall 1 \le i \le q$.

Theorem 9. *Let* $X = (X_1, \cdots, X_q)$ *be a Riesz-Dirichlet random variable with distribution* $D_{(s^1, \cdots, s^q)}$, *and let* $1 \le j \le r - 1$. *Setting*

$S_j = \sum_{l=1}^{q} (P_j^*(X_l^{-1}))^{-1}$, *we have that*

$$\left(S_j^{-1}((P_j^*(X_1^{-1}))^{-1}), \cdots, S_j^{-1}((P_j^*(X_q^{-1}))^{-1})\right)$$

has a Riesz-Dirichlet distribution $D_{(\overline{s}^1 - (r-j)\frac{d}{2}, \cdots, \overline{s}^q - (r-j)\frac{d}{2})}$, *where* $\overline{s}^i = (s_{r-j+1}^i, \cdots, s_r^i)$, $\forall 1 \le i \le q$.

Characterization 2

Let $X_1, \ldots, X_q$ be real random variables. Then $(X_1, \ldots, X_q)$ has a Dirichlet joint distribution with parameters $(p_1, \ldots, p_q)$ if and only if, for all positive real numbers $f_1, \ldots, f_q$,

$$\mathbb{E}[(\sum_{i=1}^{q} f_i X_i)^{-(p_1+\ldots+p_q)}] = \prod_{i=1}^{q} f_i^{-p_i}. \tag{1}$$

(appears in Chamayou, Letac (1994))

$P(a)(x) = axa$

For $a \in \Omega$, we define $\mathbb{H}(a)(x) = x + P(a^{\frac{1}{2}})(x)$, $x \in V$

.

Theorem 10. *Let* $p_1, \ldots, p_q$ *be in* $(\frac{r-1}{2}, +\infty)$, $p = p_1 + \ldots + p_q$ *and let* $X = (X_1, \ldots, X_q)$ *be a random variable on* $\overline{T_q}$ *with* K*-invariant distribution. Then* X *has the Dirichlet distribution with parameters* $(p_1, \ldots, p_q)$ *if and only if for all* $a_1, \ldots, a_q$ *in* Ω*, we have*

$$\mathbb{E}[(\Delta(\sum_{i=1}^{q} \mathbb{H}(a_i)(X_i)))^{-p}] = \prod_{i=1}^{q} (\Delta(\mathbb{H}(a_i)(e)))^{-p_i}. \qquad (2)$$

We set $a_i = (f_i - 1)e$, we get

$$\mathbb{E}[(\Delta(\sum_{i=1}^{q} f_i X_i))^{-p}] = \prod_{i=1}^{q} f_i^{-rp_i}.$$

Beta Riesz distribution

Theorem 11. *Let X and Y be two independent Riesz random variables $X \sim R(s,\sigma)$ and $Y \sim R(s',\sigma)$; s, $s' \in \mathbb{R}^r$ such that $s_i > \frac{i-1}{2}$, $s'_i > \frac{i-1}{2}$ for all $1 \leq i \leq r$.*
If we set $V = X + Y$ and $U = (X+Y)^{-1}(X)$, then

i) V is a Riesz random variable $V \sim R(s+s',\sigma)$ and is independent of U

ii) The density of U with respect to the Lebesgue measure is

$$\frac{1}{B_{\Omega_r}(s,s')}\Delta_{s-\frac{r+1}{2}}(x)\Delta_{s'-\frac{r+1}{2}}(e_r - x)\mathbf{1}_{\Omega_r \cap (e_r - \Omega_r)}(x),$$

where $B_{\Omega_r}(s,s')$ is the beta function defined on the symmetric cone Ω_r by

$$B_{\Omega_r}(s,s') = \frac{\Gamma_{\Omega_r}(s)\Gamma_{\Omega_r}(s')}{\Gamma_{\Omega_r}(s+s')}.$$

If X has the distribution β_{p_1,p_2}, which is K-invariant, we have

$$\mathbb{E}(\Delta_s(X)) = \frac{\Gamma_\Omega(p_1+s)}{\Gamma_\Omega(p_1)} \frac{\Gamma_\Omega(p_1+p_2)}{\Gamma_\Omega(p_1+p_2+s)}, \tag{3}$$

for all $s = (s_1, \ldots, s_r)$ such that $s_i > -(p_1 - \frac{i-1}{2})$.

Theorem 12. *Let X be a random matrix in $\Omega \cap (e - \Omega)$. Then X is $\beta_{p,q}$ distributed, where p, $q > \frac{r-1}{2}$, if and only if*

i) The distribution of X is in $\mathcal{A}$,

ii) For $i \in \{1,...,r\}$, the real random variable $\Delta_{e_i}(X)$ has the distribution $\beta_{p-\frac{i-1}{2},q}$,

iii) The $\Delta_{e_i}(X)$ are independent.

($\Rightarrow$) appears in Muirhead, as an exercise on the beta matrix $\beta_{\frac{n}{2},\frac{m}{2}}$

($\Leftarrow$) spherical Fourier transform

Olkin and Rubin Theorem

Theorem 13. *If U and V are $r \times r$ random symmetric positive definite matrices which are independently distributed, then U and V have a Wishart distribution with the same scale matrix if and only if*
** $Z = U + V$ is independent of $X = (U + V)^{-1}(U)$.*
** the distribution of X is K-invariant.*

Theorem 14. *Let U and V be two independent random variables valued in Ω_r with strictly positive twice differentiable densities. Set $Z = U + V$ and $X = P((U + V)^{-\frac{1}{2}})(U)$. If X and Z are independent then their exist $p,q \in \mathbb{R}$; $p,q > \frac{(r-1)}{2}$, and $\sigma \in \Omega_r$ such that $U \sim W_r(p,\sigma)$ and $V \sim W_r(q,\sigma)$.*

Theorem 15. *We use the division algorithm defined by the Cholesky decomposition. Let* X *and* Y *be independent random variables valued in* Ω_r *with strictly positive twice differentiable densities. Set* $V = X + Y$ *and* $U = V^{-1}(X)$*. If* U *and* V *are independent then there exist* $s,\ s' \in \mathbb{R}^r$*;* $s_i > \frac{i-1}{2}$*,* $s'_i > \frac{i-1}{2}$ *for all* $1 \leq i \leq r$*, and* $\sigma \in \Omega_r$ *such that* $X \sim R(s,\sigma)$ *and* $Y \sim R(s',\sigma)$*.*

The proof of this theorem relies on the resolution of two functional equations given in the following theorems which are interesting in their own rights.

Theorem 16. *Let* $a : \Omega \cap (e - \Omega) \longrightarrow \mathbb{R}$ *and* $f : \Omega \longrightarrow \mathbb{R}$ *be functions such that, for any* $x \in \Omega \cap (e - \Omega)$ *and* $y \in \Omega$,

$$a(x) = f(y(x)) - f(y(e_r - x))).$$

Assume that f *is differentiable, then there exist* $p \in \mathbb{R}^r$ *and* $c \in \mathbb{R}$ *such that, for any* $x \in \Omega \cap (e - \Omega)$ *and* $y \in \Omega$,

$$a(x) = \log \Delta_p(x) - \log \Delta_p(e_r - x),\ f(y) = \log \Delta_p(y) + c.$$

Theorem 17. *Let* $a_1 : \Omega \cap (e - \Omega) \longrightarrow \mathbb{R}$ *and* a_2, $f : \Omega \longrightarrow \mathbb{R}$ *be functions satisfying*

$$a_1(x) + a_2(y) = f(y(x)) + f(y(e_r - x)),$$

for any $x \in \Omega \cap (e - \Omega)$ *and* $y \in \Omega$*. Assume that* f *is twice differentiable then there exist* $p' \in \mathbb{R}^r$, $\delta \in V$ *and* c_1, c_2, $c_3 \in \mathbb{R}$ *such that for any* $x \in \Omega \cap (e - \Omega)$ *and* $y \in \Omega$*,*

$$f(y) = \log \Delta_{p'}(y) + \langle \delta, y \rangle + c_1$$

$$a_1(x) = \log \Delta_{p'}(x) + \log \Delta_{p'}(e_r - x) + c_2,$$

$$a_2(y) = 2 \log \Delta_{p'}(y) + \langle \delta, y \rangle + c_3,$$

where $2c_1 = c_2 + c_3$.

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